Identification in time series summary causal graphs

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Identification in non-temporal causal graphs

Notations:

- $\mathscr{G}$: directed acyclic graph (DAG)
- V: set of nodes in $\mathscr{G}$
- E: set of directed edges in $\mathscr{G}$
- p: number of nodes in $\mathscr{G}$

Action: Assign fixed values to a set of variables (*do* operator)

Causal query: Causal effect under the action

Estimand: Distribution of the target without the operator do

Causal query → **Estimand** → Estimation

Example: Diamond

Question: $P(V^4|do(V^2 = v^2))$?

Answer: $\sum_{v^1} P(V^1 = v^1)P(V^4|V^2 = v^2, V^1 = v^1)$

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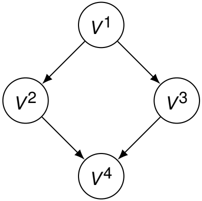

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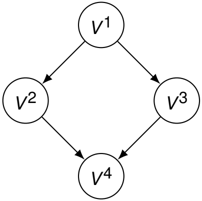

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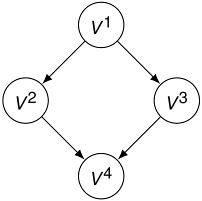

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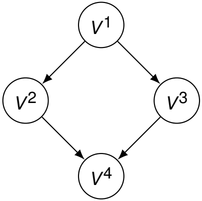

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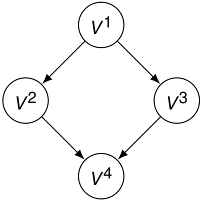

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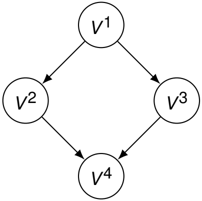

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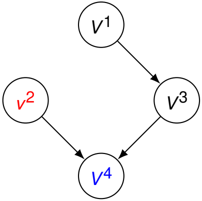

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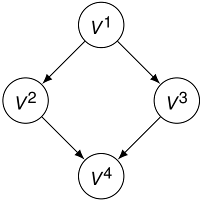

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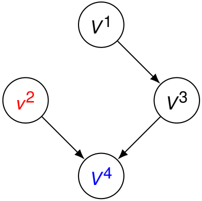

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Identification in non-temporal causal graphs

$\mathbf{X} = (X^1, X^2, X^3)$: set of time series

Full-time causal graph

Assumption 1: consistency throughout time
Assumption 2: causal sufficiency
λ_{max}: maximal lag between cause and effect

Window causal graph

C. K. Assaad, E. Devijver, E. Gaussier. Survey and Evaluation of Causal Discovery Methods for Time Series. JAIR, 2022.

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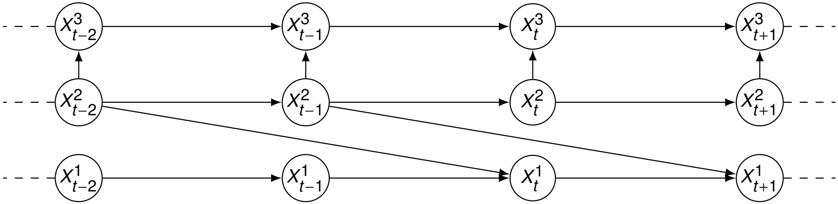

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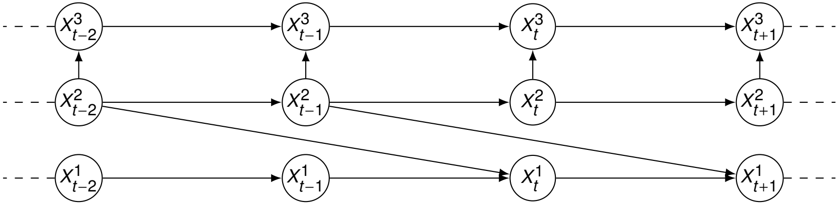

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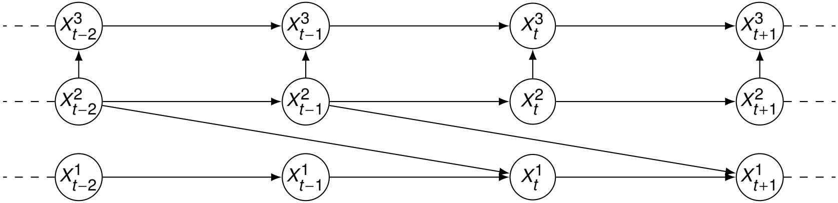

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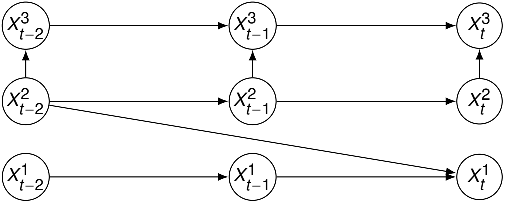

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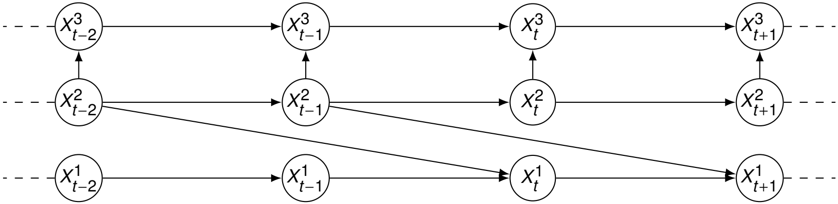

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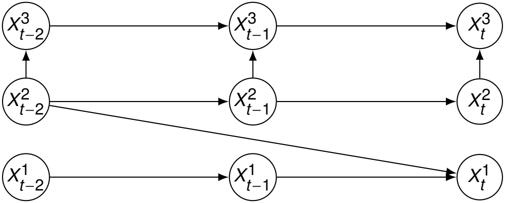
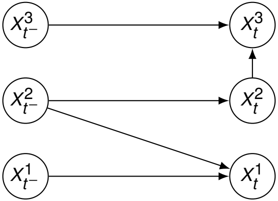


Extended summary causal graph



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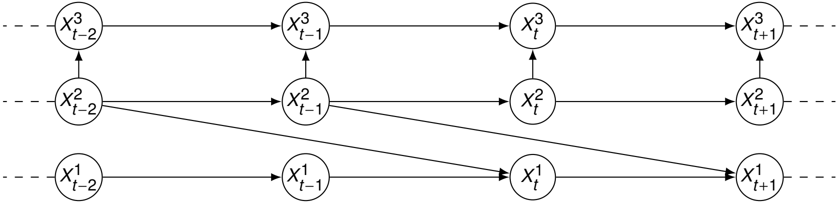

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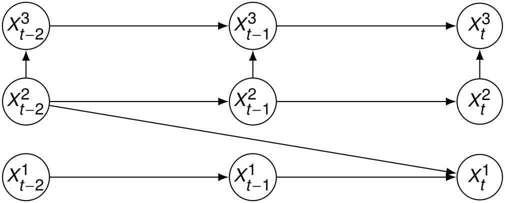

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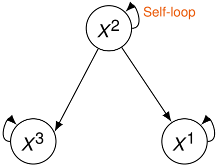

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Problem setting

Problem

Given a(n) (extended) summary graph $G_{(e)s}$, a maximal lag λ_{max} and a causal query Q, find the estimand of the causal query.

⚠ *Several estimands of the same causal query!*

Example:

- $Q = P\left(X_t^3 = x_t^3 \,|\, do(X_{t-1}^2 = x_{t-1}^2)\right)$
- $\lambda_{max} = 2$

Summary causal graph

Window causal graph 1

Window causal graph 2

$$Q = \sum_{x_{t-2}^2} P(X_{t-2}^2 = x_{t-2}^2) P\left(X_t^3 = x_t^3 \,|\, X_{t-1}^2 = x_{t-1}^2, x_{t-2}^2\right)$$

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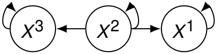

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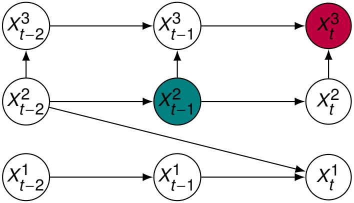
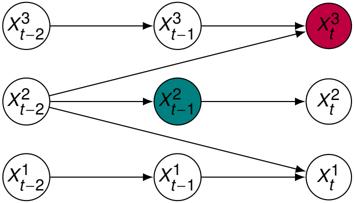


Window causal graph 2



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$$Q = P(X_t^3 = x_t^3)$$

Equivalence classes

Objective

Given a(n) (extended) summary graph G_s (or G_{es}), a maximal lag λ_{max} and a causal query

$$Q = P\left(X_t^2 = x_t^2 | do(X_{t-\mu}^1 = x_{t-\mu}^1)\right),$$

with $\mu \geq 0$, we aim to grouping together all the window graphs giving the same estimand.

Abuse of notation: $P(x_t^2 | do(x_{t-\mu}^1)) := Q$.

Definition (Equivalence classes)

Two window graphs G_{w_1} and G_{w_2} compatible with G_s (or G_{es}) are equivalent if the estimands of Q in G_{w_1} and G_{w_2} are the sames.

Technical problem: The estimand is not uniquely defined (many back-door sets).

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Equivalence classes

Idea

Each window graph G_w is characterized by the set Pa_w of X_t^2 parents in G_w.

Characterization: Each class $\mathscr{C}$ is defined by a set of parents $Pa_{\mathscr{C}}$. The estimand Q_e of Q associated with $\mathscr{C}$ is

$$Q_e = \sum_{y_{\mathscr{C}}} P(x_t^2 | x_{t-\mu}^1, y_{\mathscr{C}}) \times P(y_{\mathscr{C}}),$$

where the sum covers all potential values of $y_{\mathscr{C}}$ in $Pa_{\mathscr{C}}$.

Open question: What is the "optimal" characterization back-door set ?

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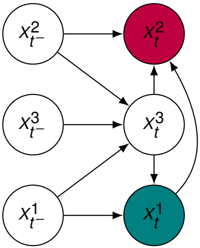
Example: Consider $\lambda_{\max} = 2$, $\mu = 0$ and

Extended summary graph



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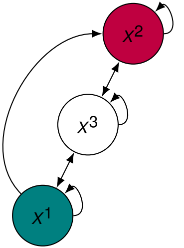

3 classes.

Classes	Parent set
$\mathscr{C}_1$	$\{X^1_{t-1}, X^3_t\}$
$\mathscr{C}_2$	$\{X^1_{t-2}, X^3_t\}$
$\mathscr{C}_3$	$\{X^1_{t-2}, X^1_{t-1}, X^3_t\}$

Number of window graphs: **243**

21 classes.

Number of window graphs: **453789**

Largest class: $\{X^1_{t-2}, X^1_{t-1}, X^3_{t-2}, X^3_{t-1}, X^3_t\}$

Equivalence classes

Back to the general case: let $\mathscr{C}_i$ and $\mathscr{C}_j$ be two different classes. Under which condition, we can have a common estimand in both classes ?

Answer: If for all window graphs G_w in $\mathscr{C}_i \cup \mathscr{C}_j$, we have

$$\left[Pa_{\mathscr{C}_i} \cup Pa_{\mathscr{C}_j}\right] \cap De(X^1_{t-\mu}, X^2_t) = \varnothing,$$

where $De(\cdot,\cdot)$ is the set of common descendants.

Proposition

Let G_{es} be an extended summary graph and $\mathscr{C}_i$, $\mathscr{C}_j$ two different classes. Then, for all G_w in $\mathscr{C}_i \cup \mathscr{C}_j$, we have

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Back to the general case: let $\mathscr{C}_i$ and $\mathscr{C}_j$ be two different classes. Under which condition, we can have a common estimand in both classes ?

Answer: If for all window graphs G_w in $\mathscr{C}_i \cup \mathscr{C}_j$, we have

$$\left[Pa_{\mathscr{C}_i} \cup Pa_{\mathscr{C}_j}\right] \cap De(X^1_{t-\mu}, X^2_t) = \varnothing,$$

where $De(\cdot,\cdot)$ is the set of common descendants.

Proposition

Let G_{es} be an extended summary graph and $\mathscr{C}_i$, $\mathscr{C}_j$ two different classes. Then, for all G_w in $\mathscr{C}_i \cup \mathscr{C}_j$, we have

$$Q_e = \sum_{y_{\mathscr{C}_{ij}}} P(x^2_t | x^1_{t-\mu}, y_{\mathscr{C}_{ij}}) \times P(y_{\mathscr{C}_{ij}}),$$

where the sum covers all potential values of $y_{\mathscr{C}_{ij}}$ in $Pa_{\mathscr{C}_i} \cup Pa_{\mathscr{C}_j}$.

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Equivalence classes

Proposition

Let G_{es} be an extended summary graph with a maximal lag $\lambda_{\max}$. Let $\mathscr{C}_{\max}$ be the largest class (with the largest cardinal). Then, for all window graph G_w, we have

$$Q_e = \sum_{y_{\mathscr{C}max}} P(x_t^2 | x_{t-\mu}^1, y_{\mathscr{C}_{\max}}) \times P(y_{\mathscr{C}_{\max}}),$$

where the sum covers all potential values of $y_{\mathscr{C}_{\max}}$ in $Pa_{\mathscr{C}_{\max}}$.

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The same result does not hold for summary graphs!

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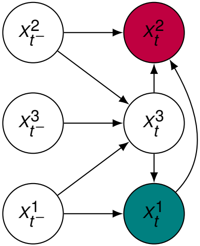
But...

The same result does not hold for summary graphs!

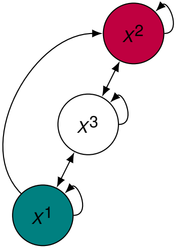
Equivalence classes

Example: Consider $\lambda_{\max} = 2$, $\mu = 0$ and

Extended summary graph



Summary graph



Universal estimand:

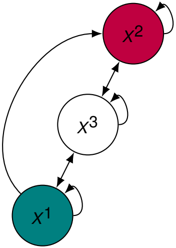
$$Q_e = \sum_{y_{\mathscr{C}max}} P(x^2_t | x^1_{t-\mu}, y_{\mathscr{C}_{\max}}) \times P(y_{\mathscr{C}_{\max}}),$$

where the sum covers all potential values of $y_{\mathscr{C}_{\max}}$ in $\left\{X^1_{t-2}, X^1_{t-1}, X^3_t\right\}$.

No universal estimand ...

Equivalence classes

No universal estimand for a summary graph.



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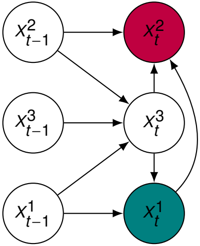
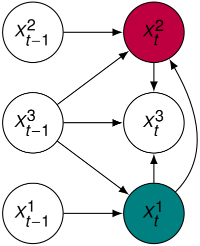


Window graph G_{w_1} in $\mathscr{C}_i$

$Pa_{\mathscr{C}_i} = \left\{X^1_{t-1}, X^3_t\right\}$



Window graph G_{w_2} in $\mathscr{C}_j$

$Pa_{\mathscr{C}_i} = \left\{X^1_{t-1}, X^3_{t-1}\right\}$

The set $Pa_{\mathscr{C}_i} \cup Pa_{\mathscr{C}_j} = \left\{X^1_{t-1}, X^3_{t-1}, X^3_t\right\}$ is not a back-door set in G_{w_2}.

The classes $\mathscr{C}_i$ and $\mathscr{C}_j$ are not super equivalent.

Equivalence classes

Definition (Super equivalence)

Let G_s be a summary graph, two classes $\mathscr{C}_i$ and $\mathscr{C}_j$ are super equivalent if $Pa_{\mathscr{C}_i} \cup Pa_{\mathscr{C}_j}$ is a valid back-door set of the query Q over $\mathscr{C}_i$ and $\mathscr{C}_j$.

Definition (Super class)

Let G_s be a summary graph, a super class is the maximal set of equivalence classes that are super equivalent for the causal query Q.

General result. If G_{es} is an extended summary graph, then there exists a unique super class. The universal estimand is

$$Q_e = \sum_{y_{\mathscr{C}}} P(x_t^2 | x_{t-\mu}^1, y_{\mathscr{C}}) \times P(y_{\mathscr{C}}),$$

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Numerical simulations

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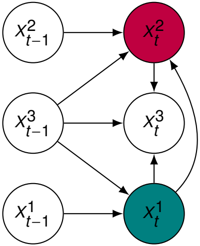
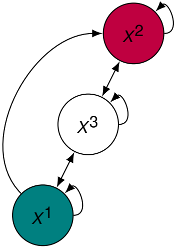


Extended summary graph



Summary graph

Generative model. Given a window causal graph G_w, the time series X^i for $i \in \{1, 2, 3\}$ is generated using the equations:

$$X^i_t = \sum_{X^j_{t'} \in Pa_{G_w}(X^i_t)} \alpha_{ij} X^j_{t'} + 0.1 \xi^i_t,$$

where $\xi^i_t \sim N(0, 1)$ and α_{ij} in $[0.2, 1]$ (randomly chosen).

We aim to estimate the expectation $\mathbb{E}[X^2_t | do(X^1_t = x^1_t)]$ for a given window graph G_w.

Numerical simulations

We show that

$$\mathbb{E}[X_t^2 | do(X_t^1 = x_t^1)] = \alpha_{21} x_t^1.$$

We estimate α_{21} using a linear regression.

Extended summary graph. Consider the classes $\mathscr{C}_1$, $\mathscr{C}_4$ and $\mathscr{C}_9$ with

$$Pa_{\mathscr{C}_1} = \left\{X_t^1, X_{t-1}^1, X_{t-1}^3\right\}, Pa_{\mathscr{C}_4} = \left\{X_t^1, X_{t-2}^1, X_{t-2}^3\right\}$$
$$Pa_{\mathscr{C}_9} = \left\{X_t^1, X_{t-1}^1, X_{t-1}^3, X_{t-2}^1, X_{t-2}^3\right\}$$

The estimation is performed by generating a 100 points for each time series.

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MAE: Mean Absolute Error.

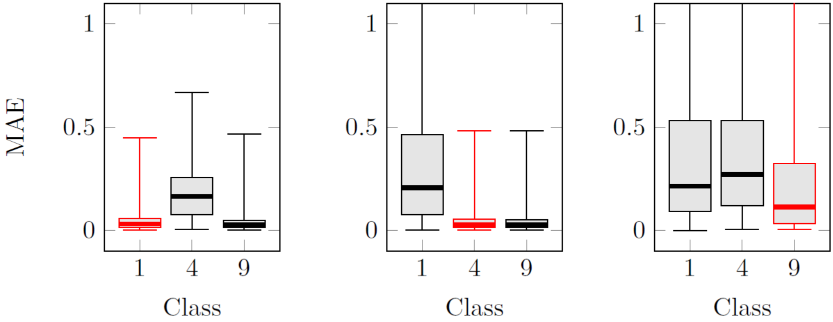
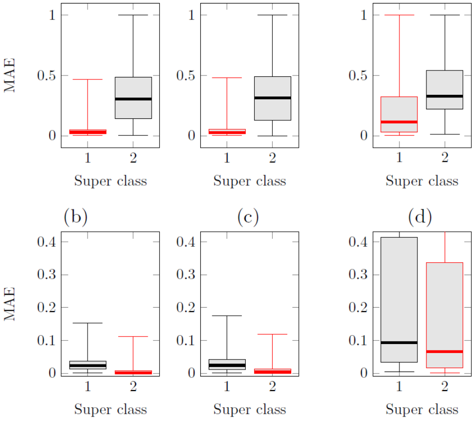


Figure: Estimation boxplots over each class.

Numerical simulations

Summary graph. Two super classes S_1 and S_2.

$$S_1 : \left\{X_t^1, X_{t-1}^1, X_{t-1}^3, X_{t-2}^1, X_{t-2}^3\right\}$$
$$S_2 : \left\{X_t^1, X_t^3, X_{t-1}^1, X_{t-1}^3, X_{t-2}^1, X_{t-2}^3\right\}$$



(b) (c) (d)

(e) (f) (g)

Practical use of the algorithm output

Algorithm output:

- Super classes $S_1, \ldots, S_p$,
- Associated estimands $Q_e^1, \ldots, Q_e^p$,
- Proportions $w_1, \ldots, w_p$.

Practical use:

- Most likely estimand:

$$MLE = \mathcal{E}_{argmax(w_1,\ldots,w_p)},$$

- Weighted mean estimand:

$$WME = \sum_{i=1}^{p} w_i Q_e^i.$$

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Conclusion and Prospect

- Searching for equivalence classes for a causal query from an extended summary or a summary graph.
- Characterizing each class of window graphs by the set of parents.
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