Causal Inference in Directed, Possibly Cyclic, Graphical Models

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Joint work with Elina Robeva [1]

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When Causal Inference meets Statistical Analysis
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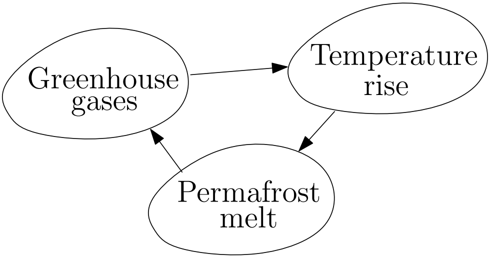

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- Our approach is an extension of the hybrid approach in the acyclic setting [Teyssie and Koller 2005 / Raskutti and Uhler 2018 / ...] to the cyclic setting.
- Previously other algorithms have been proposed which allow for cyclic graphs and have no parametric assumptions [Richardson 2013 / Hyttinen, Hoyer, Eberhardt and Jarvisalo 2013], however, they only output characteristics that are shared by all of the members of the Markov equivalence class of $G^\star$.

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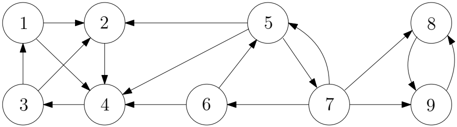

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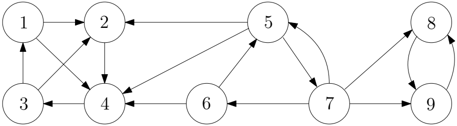

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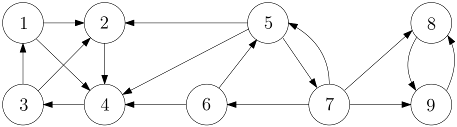

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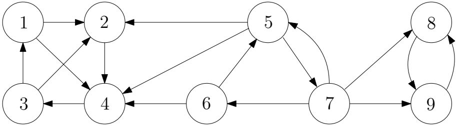

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- $(\mathcal{SC}(G^\star), \leq_{G^\star})$ is called *the partially ordered partition associated with* $G^\star$.
- We find the Markov equivalence class of $G^\star$ by optimizing a certain score over the set

$$\mathcal{S} := \left\{ (\mathcal{P}, \pi) \;\middle|\; \mathcal{P} \subseteq 2^{[n]} \text{ is a partition of } [n] \text{ and } \pi \subseteq \mathcal{P} \times \mathcal{P} \text{ is a partial order on } \mathcal{P}. \right\}.$$

Characterization of Markov Equivalence

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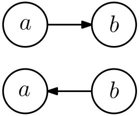

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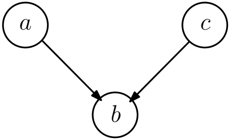

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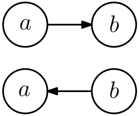

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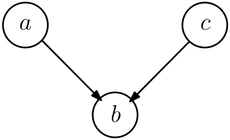

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This characterization does not hold for all directed graphs:

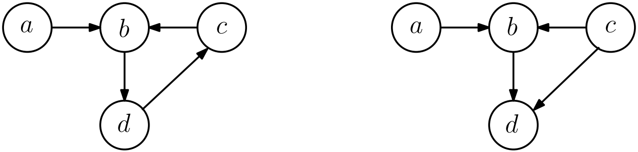

FIGURE: These two graphs have the same adjacencies and immoralities, but $a \not\perp\!\!\!\perp c$ in the first graph, and $a \perp\!\!\!\perp c$ in the second one.

Characterization of Markov Equivalence

Theorem [Richardson 1997]

Assume $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ are two directed graphs. Then G_1 and G_2 are Markov equivalent if and only if the following conditions hold:

1. G_1 and G_2 have the same p-adjacencies.
2. G_1 and G_2 have the same set of unshielded non-conductors.
3. G_1 and G_2 have the same set of unshielded imperfect non-conductors.
4. If (a, b_1, c) and (a, b_2, c) are unshielded imperfect non-conductors (in G_1 and G_2), then b_1 is an ancestor of b_2 in G_1 if and only if b_1 is an ancestor of b_2 in G_2.
5. For any $t \in \mathbb{N}$, triples (a_0, a_1, a_2) and (a_{t-1}, a_t, a_{t+1}) are mutually exclusive with respect to the uncovered itinerary $P = (a_0, a_1, a_2, \ldots, a_{t-1}, a_t, a_{t+1})$ in G_1 if and only if (a_0, a_1, a_2) and (a_{t-1}, a_t, a_{t+1}) are mutually exclusive with respect to P in G_2.
6. If (a_0, a_1, a_2) and (a_{t-1}, a_t, a_{t+1}) are mutually exclusive with respect to some uncovered itinerary $(a_0, a_1, \ldots, a_{t+1})$ and (a_0, b_1, a_{t+1}) is an unshielded imperfect non-conductor (in G_1 and G_2), then a_1 is an ancestor of b in G_1 if and only if a_1 is an ancestor of b in G_2.

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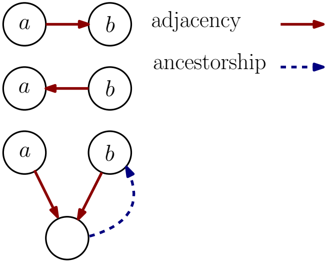

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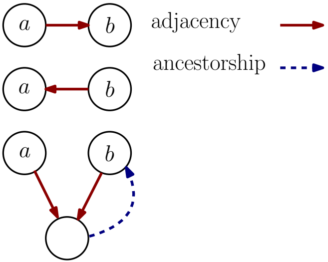

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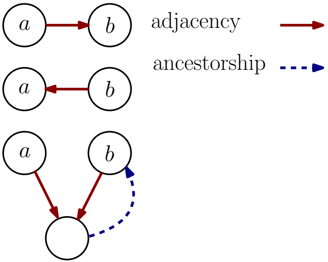

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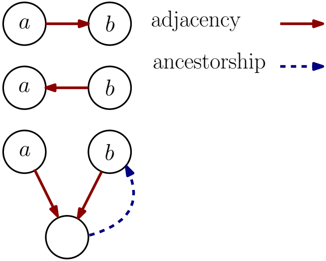

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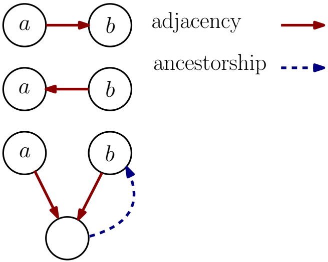

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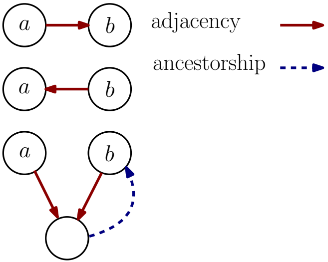

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- a and b have a common child in G, which is an ancestor of a or b.

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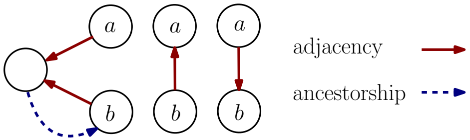

Figure: P-adjacency

Proposition

For each $(\mathcal{P}, \pi) \in \mathcal{S}$, let

$$E^{(1)}_{(\mathcal{P},\pi)} := \left\{ (a,b) \in [n]^2 \;\middle|\; a \neq b,\ a \not\perp b \mid \bigcup \{ C \in \mathcal{P} \mid C \leq_\pi \max\{C_{a,\mathcal{P}}, C_{b,\mathcal{P}}\} \} \setminus \{a,b\} \right\}.$$

Also, define $S_1 := \operatorname{argmin}_{(\mathcal{P},\pi)\in\mathcal{S}} \left| E^{(1)}_{(\mathcal{P},\pi)} \right|$. Then

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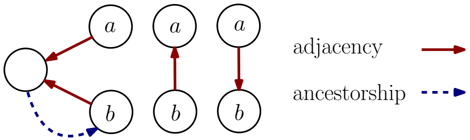

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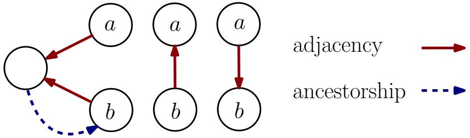

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- for every $(\mathcal{P}, \pi) \in S_1$, the set $E^{(1)}_{(\mathcal{P},\pi)}$ is equal to the set of p-adjacencies in $G^\star$.

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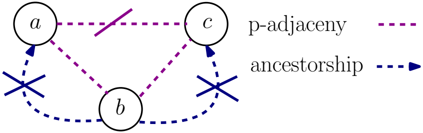

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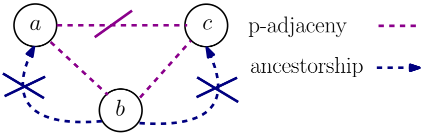

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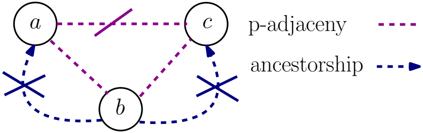

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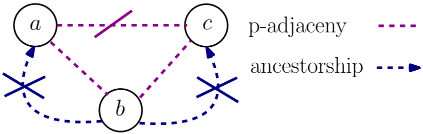

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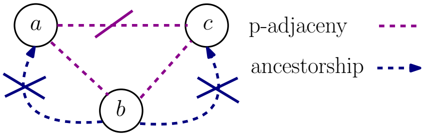

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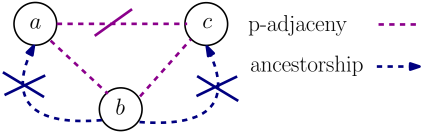

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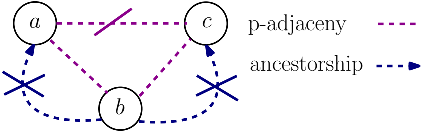

FIGURE: Unshielded non-conductor

PROPOSITION

For each $(\mathcal{P}, \pi) \in \mathcal{S}$, let

$$E^{(2)}_{(\mathcal{P},\pi)} := \Big\{(a,b,c) \in [n]^3 \mid a,b,c \text{ are distinct, } (a,b),(c,b) \in E^{(1)}_{(\mathcal{P},\pi)},\ (a,c) \notin E^{(1)}_{(\mathcal{P},\pi)},\ C_{b,\mathcal{P}} \not\leq_\pi C_{a,\mathcal{P}},\ C_{b,\mathcal{P}} \not\leq_\pi C_{c,\mathcal{P}}\Big\}.$$

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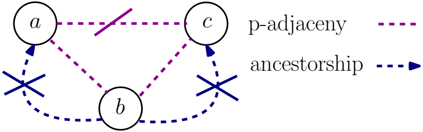

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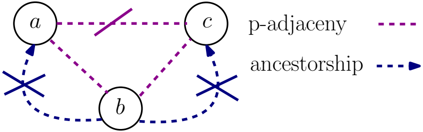

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- the partially ordered partition associated with $G^\star$ is in S_2, and
- for every $(\mathcal{P}, \pi) \in S_2$, the set $E^{(2)}_{(\mathcal{P},\pi)}$ is equal to the set of unshielded non-conductors in $G^\star$.

Other ingredients for defining the score

$$E^{(3)}_{(\mathcal{P},\pi)} := \Big\{(a,b,c) \in [n]^3 \mid (a,b,c) \in E^{(2)}_{(\mathcal{P},\pi)},$$
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For all $t \in [n-2]$,

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$$(a_i,a_{i-1}) \in E^{(1)}_{(\mathcal{P},\pi)} \forall\, i \in [t+1],$$
$$(a_i,a_j) \notin E^{(1)}_{(\mathcal{P},\pi)} \forall\, i \in \{2,\ldots,t+1\}, j \leq i-2,$$
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$$E^{(6)}_{(\mathcal{P},\pi)} := \bigcup_{t=1}^{n-2} \Big\{ (a_0, a_1, \ldots, a_t, a_{t+1}, a_0, b, a_{t+1}) \in [n]^{t+5} \mid (a_0, b, a_{t+1}) \in E^{(3)}_{(\mathcal{P},\pi)}, \\ (a_0, a_1, \ldots, a_t, a_{t+1}) \in D^{(t)}_{(\mathcal{P},\pi)}, \\ C_{a_1,\mathcal{P}} \leq_\pi C_{b,\mathcal{P}} \Big\}.$$

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lexicographical order: For $x, y \in \mathbb{Z}^{n+2}$,

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THEOREM

Each minimizer of the graphical score over $\mathcal{S}$, uniquely determines the Markov equivalence class of $G^\star$.

Outline

Graph of the partially ordered partitions

Consider the graph $\mathcal{G} = (\mathcal{S}, \mathcal{E})$, where $((\mathcal{P}_1, \pi_1), (\mathcal{P}_2, \pi_2)) \in \mathcal{E}$ iff $\mathrm{GS}(\mathcal{P}_1, \pi_1) \geq \mathrm{GS}(\mathcal{P}_2, \pi_2)$ and

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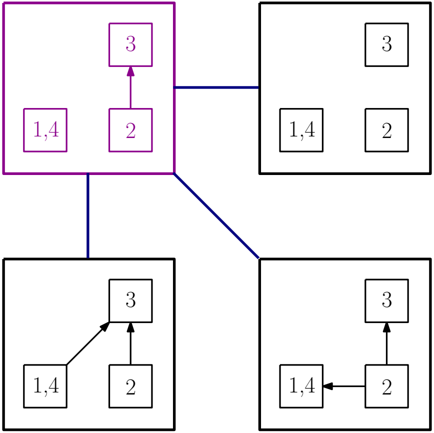

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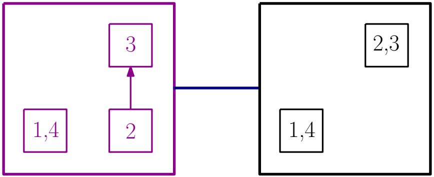

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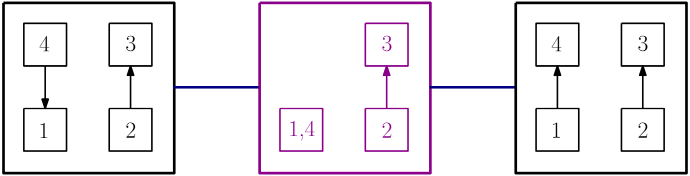

Based on experimental evidence, we conjecture that for any $(\mathcal{P}, \pi) \in \mathcal{S}$, there exist an optimal partially ordered partition $(\mathcal{P}_0, \pi_0)$ and a directed path in $\mathcal{G}$ such that the path starts from $(\mathcal{P}, \pi)$ and ends at $(\mathcal{P}_0, \pi_0)$.

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Algorithm 1 Markov equivalence class discovery

Input: The set of all the conditional independence statements satisfied by distribution $\mathbb{P}$ and an initial $(\mathcal{P}_1, \pi_1) \in \mathcal{S}$.

Output: An optimal partially ordered partition.

1: Set $\hat{\mathcal{P}} := \mathcal{P}_1$ and $\hat{\pi} := \pi_1$.
2: Perform a depth-first search on $\mathcal{G}$ with root $(\hat{\mathcal{P}}, \hat{\pi})$ to find a directed path from $(\hat{\mathcal{P}}, \hat{\pi})$ to a partially ordered partition $(\tilde{\mathcal{P}}, \tilde{\pi})$ with $\mathrm{GS}(\hat{\mathcal{P}}, \hat{\pi}) > \mathrm{GS}(\tilde{\mathcal{P}}, \tilde{\pi})$.
3: **if** $(\tilde{\mathcal{P}}, \tilde{\pi})$ is found **then**
4: Set $\hat{\mathcal{P}} := \tilde{\mathcal{P}}$ and $\hat{\pi} := \tilde{\pi}$, and go back to step 2.
5: **else**
6: **return** $(\hat{\mathcal{P}}, \hat{\pi})$.
7: **end if**

- Once this algorithm finds an optimal partially ordered partition $(\hat{\mathcal{P}}, \hat{\pi})$, it needs to perform a full depth-first search on $\mathcal{G}$ with root $(\hat{\mathcal{P}}, \hat{\pi})$ before it makes sure that there is no partially ordered partition with a lower score.

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- In practice, we perform a greedy version of this algorithm, where we stop the depth-first search as soon as a directed path of length N consisting of partially ordered partitions with the same score is observed, where N is a threshold given to the algorithm as an input.

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- In practice, we perform a greedy version of this algorithm, where we stop the depth-first search as soon as a directed path of length N consisting of partially ordered partitions with the same score is observed, where N is a threshold given to the algorithm as an input.
- One can choose to repeat the algorithm M times with M different initial partially ordered partitions to get better results, where M is also part of the input.

- Random graphs were generated according to the Erdős–Rényi model and using the `R` library `igraph`. The set of all d-separations satisfied by each graph was then generated and given to this algorithm as part of its input.

Simulations

- Random graphs were generated according to the Erdős–Rényi model and using the `R` library `igraph`. The set of all d-separations satisfied by each graph was then generated and given to this algorithm as part of its input.
- We used the following initial partially ordered partitions ($M = 3$):

$$\mathcal{P}_1 = \{\{1, \ldots, n\}\},$$
$$\mathcal{P}_2 = \left\{\left\{1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}, \left\{\left\lfloor \frac{n}{2} \right\rfloor + 1, \ldots, n\right\}\right\}, \left\{1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\} \leq_{\pi_2} \left\{\left\lfloor \frac{n}{2} \right\rfloor + 1, \ldots, n\right\},$$
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- The positive integers N used in our experiments were the following:

number of vertices of the graph (n)	N
7	30
8	30
9	40
10	50

SIMULATIONS

FIGURE: Results of running the greedy optimization algorithm on 120 random graphs with 7 vertices and different levels of sparsity. Thirty graphs were tested for each of the probabilities 0.2, 0.4, 0.6, and 0.8. From left to right, the success rates are 0.93, 0.97, 1 and 1.

Simulations

FIGURE: Results of running the greedy optimization algorithm on 120 random graphs with different numbers of vertices n where each edge appears in the graph with probability 0.3. Thirty graphs were tested for each n. From left to right, the success rates are 0.90, 0.93, 0.97 and 0.93.

Outline

THEOREM

Suppose that $(\mathcal{P}, \pi)$ is an optimal partially ordered partition, and $G = ([n], E)$ is a directed graph with the following properties:

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1. For all $(a, b) \in [n]^2$, a and b are p-adjacent in G if and only if $(a, b) \in E^{(1)}_{(\mathcal{P},\pi)}$.

Theorem

Suppose that $(\mathcal{P}, \pi)$ is an optimal partially ordered partition, and $G = ([n], E)$ is a directed graph with the following properties:

1. For all $(a, b) \in [n]^2$, a and b are p-adjacent in G if and only if $(a, b) \in E^{(1)}_{(\mathcal{P},\pi)}$.
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3. If $(a, b, c) \in E^{(2)}_{(\mathcal{P},\pi)} \setminus E^{(3)}_{(\mathcal{P},\pi)}$ and for all $b' \in [n]$ with $C_{b',\mathcal{P}} <_\pi C_{b,\mathcal{P}}$ and $(a, b', c) \in E^{(2)}_{(\mathcal{P},\pi)}$, $(a, b', c) \in E^{(3)}_{(\mathcal{P},\pi)}$, then a and c have a common child in $C_{b,\mathcal{P}}$ in graph G.

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4. If $(a, b, c) \in E^{(3)}_{(\mathcal{P},\pi)}$, then a and c don't have a common child in $C_{b,\mathcal{P}}$ in graph G.

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3. If $(a, b, c) \in E^{(2)}_{(\mathcal{P},\pi)} \setminus E^{(3)}_{(\mathcal{P},\pi)}$ and for all $b' \in [n]$ with $C_{b',\mathcal{P}} <_\pi C_{b,\mathcal{P}}$ and $(a, b', c) \in E^{(2)}_{(\mathcal{P},\pi)}$, $(a, b', c) \in E^{(3)}_{(\mathcal{P},\pi)}$, then a and c have a common child in $C_{b,\mathcal{P}}$ in graph G.
4. If $(a, b, c) \in E^{(3)}_{(\mathcal{P},\pi)}$, then a and c don't have a common child in $C_{b,\mathcal{P}}$ in graph G.

Then G is Markov equivalent to $G^\star$.

Moreover, $G^\star$ satisfies properties 1 to 4 with respect to the partially ordered partition associated with $G^\star$.

RECOVERING A GRAPH FROM A PARTIALLY ORDERED PARTITION

Algorithm 3 Markov equivalent graph discovery

Input: The set of all the conditional independence statements satisfied by distribution $\mathbb{P}$, an optimal partially ordered partition $(\mathcal{P}, \pi)$.

Output: A graph Markov equivalent to $G^\star$.

1: Set $\hat{\mathcal{P}} := \mathcal{P}$ and $\hat{\pi} := \pi$.
2: **if** there exist $a, b \in [n]$ such that $(a,b) \in E^{(1)}_{(\hat{\mathcal{P}},\hat{\pi})}$, but $C_{a,\hat{\mathcal{P}}} \not\leq_{\hat{\pi}} C_{b,\hat{\mathcal{P}}}$ and $C_{b,\hat{\mathcal{P}}} \not\leq_{\hat{\pi}} C_{a,\hat{\mathcal{P}}}$ **then**
3:     Go to step 19.
4: **end if**
5: **if** for a consecutive pair (C_1, C_2) in $(\hat{\mathcal{P}}, \hat{\pi})$ and for all $a \in C_1, b \in C_2$, $(a,b) \notin E^{(1)}_{(\hat{\mathcal{P}},\hat{\pi})}$ **then**
6:     Go to step 19.
7: **end if**
8: Consider a linear extension $\hat{\mathcal{P}} = \{C_1, \dots, C_{|\hat{\mathcal{P}}|}\}$ of the partial order $\hat{\pi}$.
9: **for** i in $1 : |\hat{\mathcal{P}}|$ **do**
10:     Set $A_{C_i} := \left\{ (a,b) \in E^{(1)}_{(\hat{\mathcal{P}},\hat{\pi})} \;\middle|\; a, b \in C_i \right\}$.
11:     Set $B_{C_i} := \left\{ (a,b) \in E^{(1)}_{(\hat{\mathcal{P}},\hat{\pi})} \;\middle|\; a \in \bigcup_{j=1}^{i-1} C_j, b \in C_i \right\}$.
12:     Set $ComCh_{C_i}$ and $NoComCh_{C_i}$ to be the sets of pairs that must have and must not have common children in C_i per conditions 3 and 4 of the previous theorem respectively.
13:     Set $E_{C_i} := \mathrm{SCCR}\left(C_i, A_{C_i}, B_{C_i}, ComCh_{C_i}, NoComCh_{C_i}\right)$.
14:     **if** no E_{C_i} is outputted **then**
15:         Go to step 19.
16:     **end if**
17: **end for**
18: **return** $G = \left([n], \bigcup_{i=1}^{|\hat{\mathcal{P}}|} E_{C_i}\right)$.
19: Perform a depth-first search on $\mathcal{G}$ with root $(\hat{\mathcal{P}}, \hat{\pi})$ to find a different partially ordered partition $(\tilde{\mathcal{P}}, \tilde{\pi})$ with $\mathrm{GS}(\hat{\mathcal{P}}, \hat{\pi}) = \mathrm{GS}(\tilde{\mathcal{P}}, \tilde{\pi})$. Set $\hat{\mathcal{P}} := \tilde{\mathcal{P}}$ and $\hat{\pi} := \tilde{\pi}$, and go back to step 2.

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Input: The set of all the conditional independence statements satisfied by distribution $\mathbb{P}$, an optimal partially ordered partition $(\mathcal{P}, \pi)$.

Output: A graph Markov equivalent to $G^\star$.

1: Set $\hat{\mathcal{P}} \coloneqq \mathcal{P}$ and $\hat{\pi} \coloneqq \pi$.
2: **if** there exist $a, b \in [n]$ such that $(a,b) \in E^{(1)}_{(\hat{\mathcal{P}},\hat{\pi})}$, but $C_{a,\hat{\mathcal{P}}} \not\leq_{\hat{\pi}} C_{b,\hat{\mathcal{P}}}$ and $C_{b,\hat{\mathcal{P}}} \not\leq_{\hat{\pi}} C_{a,\hat{\mathcal{P}}}$ **then**
3: Go to step 19.
4: **end if**
5: **if** for a consecutive pair (C_1, C_2) in $(\hat{\mathcal{P}}, \hat{\pi})$ and for all $a \in C_1, b \in C_2$, $(a,b) \notin E^{(1)}_{(\hat{\mathcal{P}},\hat{\pi})}$ **then**
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19: Perform a depth-first search on $\mathcal{G}$ with root $(\hat{\mathcal{P}}, \hat{\pi})$ to find a different partially ordered partition $(\tilde{\mathcal{P}}, \tilde{\pi})$ with $\mathrm{GS}(\hat{\mathcal{P}}, \hat{\pi}) = \mathrm{GS}(\tilde{\mathcal{P}}, \tilde{\pi})$. Set $\hat{\mathcal{P}} \coloneqq \tilde{\mathcal{P}}$ and $\hat{\pi} \coloneqq \tilde{\pi}$, and go back to step 2.

Algorithm 3 Markov equivalent graph discovery

Input: The set of all the conditional independence statements satisfied by distribution $\mathbb{P}$, an optimal partially ordered partition $(\hat{\mathcal{P}}, \hat{\pi})$

The SCCR algorithm determines whether a directed graph $G_{C_i} = ([n], E_{C_i})$ with the following properties exists and outputs E_{C_i} if so:

12: Set $ComCh_{C_i}$ and $NoComCh_{C_i}$ to be the sets of pairs that must have and must not have common children in C_i per conditions 3 and 4 of the previous theorem respectively.

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The SCCR algorithm determines whether a directed graph $G_{C_i} = ([n], E_{C_i})$ with the following properties exists and outputs E_{C_i} if so:

- For any $a, b \in [n]$, a and b are p-adjacent in G_{C_i} iff $(a, b) \in A_{C_i} \cup B_{C_i}$ or $(b, a) \in A_{C_i} \cup B_{C_i}$,

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- C_i is a strongly connected component in G_{C_i}, and $C_i \not\leq_{G_{C_i}} C_{a,G_{C_i}}$ for all $a \in [n] \setminus C_i$,

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Algorithm 3 Markov equivalent graph discovery

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- C_i is a strongly connected component in G_{C_i}, and $C_i \not\leq_{G_{C_i}} C_{a,G_{C_i}}$ for all $a \in [n] \setminus C_i$,
- for any $(a, b) \in ComCh_{C_i}$, a and b have a common child in C_i, and
- for any $(a, b) \in NoComCh_{C_i}$, a and b do not have a common child in C_i.

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Recovering a graph from a partially ordered partition

Algorithm 3 Markov equivalent graph discovery

Input: The set of all the conditional independence statements satisfied by distribution $\mathbb{P}$, an optimal partially ordered partition $(\mathcal{P}, \pi)$.

Output: A graph Markov equivalent to $G^\star$.

1: Set $\hat{\mathcal{P}} := \mathcal{P}$ and $\hat{\pi} := \pi$.
2: **if** there exist $a, b \in [n]$ such that $(a,b) \in E^{(1)}_{(\hat{\mathcal{P}},\hat{\pi})}$, but $C_{a,\hat{\mathcal{P}}} \not\leq_{\hat{\pi}} C_{b,\hat{\mathcal{P}}}$ and $C_{b,\hat{\mathcal{P}}} \not\leq_{\hat{\pi}} C_{a,\hat{\mathcal{P}}}$ **then**
3:     Go to step 19.
4: **end if**
5: **if** for a consecutive pair (C_1, C_2) in $(\hat{\mathcal{P}}, \hat{\pi})$ and for all $a \in C_1, b \in C_2$, $(a,b) \notin E^{(1)}_{(\hat{\mathcal{P}},\hat{\pi})}$ **then**
6:     Go to step 19.
7: **end if**
8: Consider a linear extension $\hat{\mathcal{P}} = \{C_1, \ldots, C_{|\hat{\mathcal{P}}|}\}$ of the partial order $\hat{\pi}$.
9: **for** i in $1 : |\hat{\mathcal{P}}|$ **do**
10:     Set $A_{C_i} := \left\{ (a,b) \in E^{(1)}_{(\hat{\mathcal{P}},\hat{\pi})} \;\middle|\; a, b \in C_i \right\}$.
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14:     **if** no E_{C_i} is outputted **then**
15:         Go to step 19.
16:     **end if**
17: **end for**
18: **return** $G = \left([n], \bigcup_{i=1}^{|\hat{\mathcal{P}}|} E_{C_i}\right)$.
19: Perform a depth-first search on $\mathcal{G}$ with root $(\hat{\mathcal{P}}, \hat{\pi})$ to find a different partially ordered partition $(\tilde{\mathcal{P}}, \tilde{\pi})$ with $\mathrm{GS}(\hat{\mathcal{P}}, \hat{\pi}) = \mathrm{GS}(\tilde{\mathcal{P}}, \tilde{\pi})$. Set $\hat{\mathcal{P}} := \tilde{\mathcal{P}}$ and $\hat{\pi} := \tilde{\pi}$, and go back to step 2.

SCCR ALGORITHM SIMULATIONS

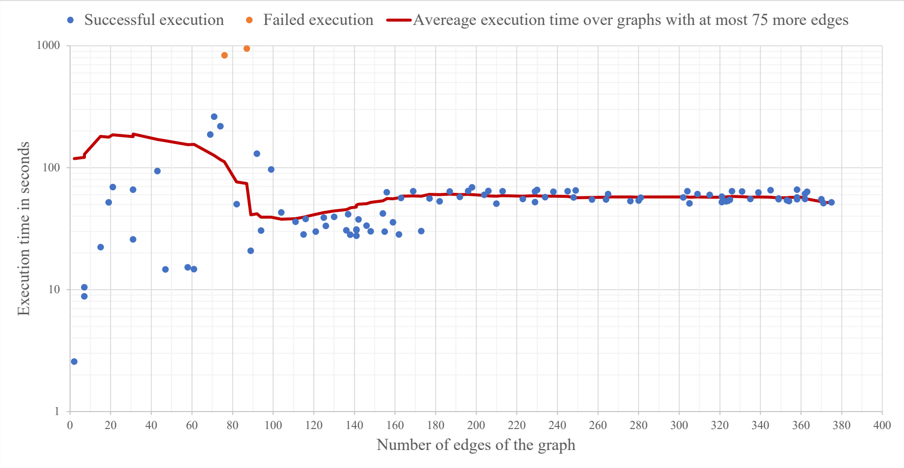

FIGURE: Results of running the SCCR algorithm with $N = 100$ on the partially ordered partitions associated with 100 random graphs with 20 vertices generated according to the Erdős–Rényi model. The success rate is 0.98 and the average execution time is 71.90 seconds.

- Random graphs were generated according to the Erdős–Rényi model and using the `R` library `igraph`. The set of all d-separations satisfied by each graph was then generated and given to this algorithm as its input.

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- We used the greedy optimization algorithm with the same parameters as before.
- We used the SCCR algorithm with $N = 100$.
- In the Markov equivalent graph discovery algorithm, we restricted ourselves to testing at most 300 optimal partially ordered partitions.

FIGURE: Results of running the Markov equivalent graph discovery algorithm on 120 random graphs with 7 vertices and different levels of sparsity. Thirty graphs were tested for each of the probabilities 0.2, 0.4, 0.6, and 0.8. From left to right, the success rates are 0.3, 0.97, 1 and 1.

FIGURE: Results of running the Markov equivalent graph discovery algorithm on 120 random graphs with different numbers of vertices n where each edge appears in the graph with probability 0.3. Thirty graphs were tested for each n. From left to right, the success rates are 0.77, 0.90, 0.90 and 0.93.

Thank you for listening!

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Thomas Richardson.
A Discovery Algorithm for Directed Cyclic Graphs.

Outline

Directed cycles in the causal graph of an SCM

Assume that we study the investment and consumption in a closed market for a period of 3 (or more) years, where the initial consumption (corresponding to year 0) is assumed to be a constant such as u_0. [Haavelmo, 1943]

Directed cycles in the causal graph of an SCM

Assume that we study the investment and consumption in a closed market for a period of 3 (or more) years, where the initial consumption (corresponding to year 0) is assumed to be a constant such as u_0. [Haavelmo, 1943]

- The propensity to consume: If the group of all consumers in the society are provided with the total income r_t in year t, then they will spend a total amount u_t for consumption in that year, equal to

$$u_t = \alpha \cdot r_t + \beta + \varepsilon_t^{(1)},$$

  where α and β are constants, and $\varepsilon_t^{(1)}$ is a noise random variable.

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- The propensity to invest: If the group of all (private) investors in the society are repeatedly confronted with an increase, δ_t, over the year t, in the consumption of goods, they will invest an amount v_t in the year t, given by

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where κ is a constant, and $\varepsilon_t^{(2)}$ is a noise random variable.

- Closed market identity: In a closed market, all the income will either be spent on consumption goods or invested. So,

$$r_t = u_t + v_t.$$

Directed cycles in the causal graph of an SCM

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$$
\begin{aligned}
u_1 &= \alpha \cdot r_1 + \beta + \varepsilon_1^{(1)}, & u_2 &= \alpha \cdot r_2 + \beta + \varepsilon_2^{(1)}, & u_3 &= \alpha \cdot r_3 + \beta + \varepsilon_3^{(1)}, \\
v_1 &= \kappa \cdot (u_1 - u_0) + \varepsilon_1^{(2)}, & v_2 &= \kappa \cdot (u_2 - u_1) + \varepsilon_2^{(2)}, & v_3 &= \kappa \cdot (u_3 - u_2) + \varepsilon_3^{(2)}, \\
r_1 &= u_1 + v_1, & r_2 &= u_2 + v_2, & r_3 &= u_3 + v_3.
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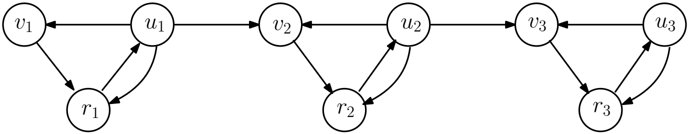
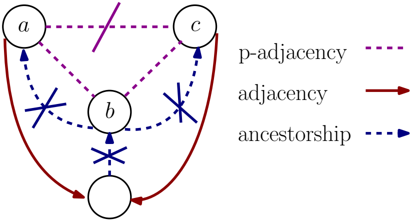


Outline

G_1 and G_2 have the same set of unshielded imperfect non-conductors.

Condition 3

G_1 and G_2 have the same set of unshielded imperfect non-conductors.



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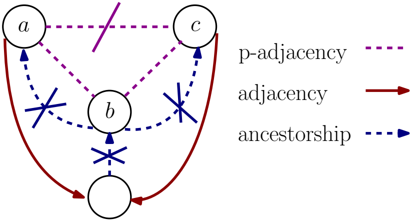

Let $G = (V, E)$ be a directed graph. A triple $(a, b, c) \in V^3$ is said to be an *unshielded imperfect non-conductor* if

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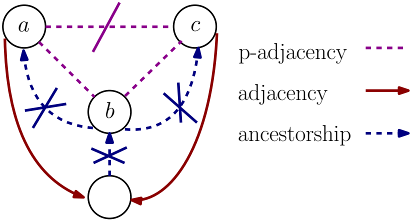

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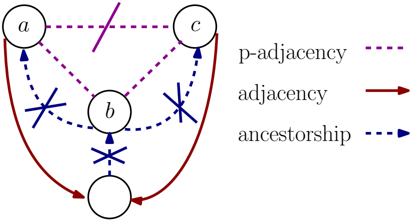

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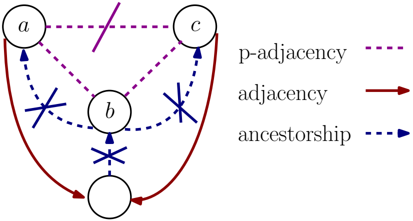

Let $G = (V, E)$ be a directed graph. A triple $(a, b, c) \in V^3$ is said to be an *unshielded imperfect non-conductor* if

- (a, b, c) is an unshielded non-conductor, and
- b is not the descendant of any of the common children of a and c.

Condition 3

G_1 and G_2 have the same set of unshielded imperfect non-conductors.

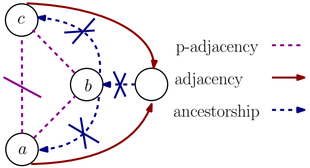

Figure: Unshielded imperfect non-conductor

Proposition

For each $(\mathcal{P}, \pi) \in \mathcal{S}$, let

$$E^{(3)}_{(\mathcal{P},\pi)} := \Big\{(a,b,c) \in [n]^3 \mid (a,b,c) \in E^{(2)}_{(\mathcal{P},\pi)},$$
$$a \perp\!\!\!\perp c \mid \bigcup \Big\{ C \in \mathcal{P} \;\Big|\; C \leq_\pi \max\{C_{a,\mathcal{P}}, C_{b,\mathcal{P}}, C_{c,\mathcal{P}}\}\Big\} \setminus \{a,c\}\Big\}.$$

Also, define $S_3 := \operatorname{argmax}_{(\mathcal{P},\pi)\in S_2} \left|E^{(3)}_{(\mathcal{P},\pi)}\right|$. Then

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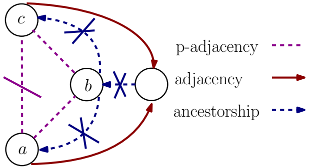

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Also, define $S_3 := \operatorname{argmax}_{(\mathcal{P},\pi)\in S_2} \left|E^{(3)}_{(\mathcal{P},\pi)}\right|$. Then

- the partially ordered partition associated with $G^\star$ is in S_3, and

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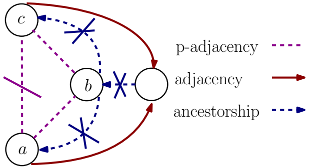

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- the partially ordered partition associated with $G^\star$ is in S_3, and
- for every $(\mathcal{P}, \pi) \in S_3$, the set $E^{(3)}_{(\mathcal{P},\pi)}$ is equal to the set of unshielded imperfect non-conductors in $G^\star$.

If (a, b_1, c) and (a, b_2, c) are unshielded imperfect non-conductors (in G_1 and G_2), then b_1 is an ancestor of b_2 in G_1 iff b_1 is an ancestor of b_2 in G_2.

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Proposition

For every $(\mathcal{P}, \pi) \in \mathcal{S}$, let

$$E^{(4)}_{(\mathcal{P},\pi)} := \{ ((a, b_1, c), (a, b_2, c)) \in [n]^3 \times [n]^3 \mid (a, b_1, c), (a, b_2, c) \in E^{(3)}_{(\mathcal{P},\pi)} \\ C_{b_1,\mathcal{P}} \leq_\pi C_{b_2,\mathcal{P}} \}.$$

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- the partially ordered partition associated with $G^\star$ is in S_4, and

Condition 4

If (a, b_1, c) and (a, b_2, c) are unshielded imperfect non-conductors (in G_1 and G_2), then b_1 is an ancestor of b_2 in G_1 iff b_1 is an ancestor of b_2 in G_2.

Proposition

For every $(\mathcal{P}, \pi) \in \mathcal{S}$, let

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Also, define $S_4 := \operatorname{argmin}_{(\mathcal{P},\pi)\in S_3} \left| E^{(4)}_{(\mathcal{P},\pi)} \right|$. Then

- the partially ordered partition associated with $G^\star$ is in S_4, and
- for every $(\mathcal{P}, \pi) \in S_4$, $((a, b_1, c), (a, b_2, c)) \in E^{(4)}_{(\mathcal{P},\pi)}$ if and only if $(a, b_1, c), (a, b_2, c)$ are unshielded imperfect non-conductors and b_1 is an ancestor of b_2 in $G^\star$.

Condition 5

For any $t \in \mathbb{N}$, triples (a_0, a_1, a_2) and (a_{t-1}, a_t, a_{t+1}) are mutually exclusive with respect to the uncovered itinerary $P = (a_0, a_1, a_2, \ldots, a_{t-1}, a_t, a_{t+1})$ in G_1 iff (a_0, a_1, a_2) and (a_{t-1}, a_t, a_{t+1}) are mutually exclusive with respect to P in G_2.

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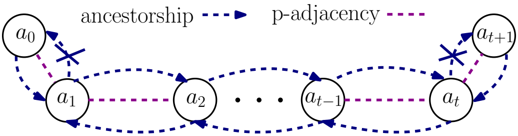

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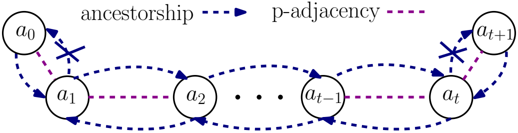

Let $G = (V, E)$ be a directed graph. For $t \in \mathbb{N}$ and $a_0, a_1, \ldots, a_t, a_{t+1} \in V$, the triples $(a_0, a_1, a_2), (a_{t-1}, a_t, a_{t+1})$ are said to be *mutually exclusive with respect to an uncovered itinerary* $(a_0, a_1, a_2, \ldots, a_{t-1}, a_t, a_{t+1})$ if

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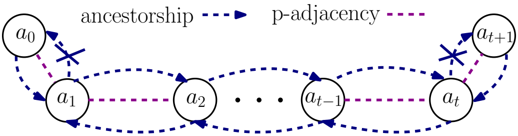

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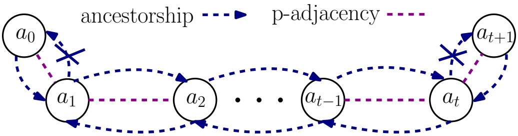

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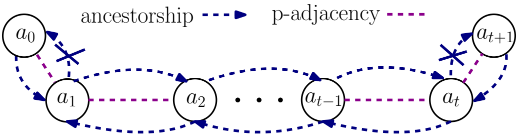

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- $a_1, \ldots, a_t$ are all in the same strongly connected component of G,

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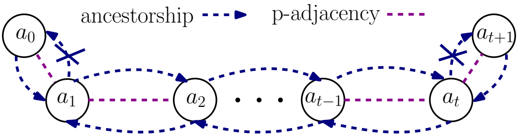
For any $t \in \mathbb{N}$, triples (a_0, a_1, a_2) and (a_{t-1}, a_t, a_{t+1}) are mutually exclusive with respect to the uncovered itinerary $P = (a_0, a_1, a_2, \ldots, a_{t-1}, a_t, a_{t+1})$ in G_1 iff (a_0, a_1, a_2) and (a_{t-1}, a_t, a_{t+1}) are mutually exclusive with respect to P in G_2.



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- a_0 and a_{t+1} are not in this component, and

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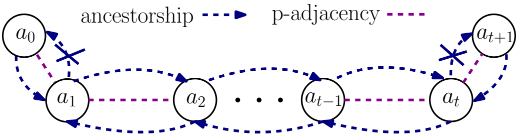

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- $a_1, \ldots, a_t$ are all in the same strongly connected component of G,
- a_0 and a_{t+1} are not in this component, and
- a_0 and a_{t+1} are ancestors of a_1.

Condition 5

Proposition

For all $t \in [n-2]$ and $(\mathcal{P}, \pi) \in \mathcal{S}$, let

$$D^{(t)}_{(\mathcal{P},\pi)} \coloneqq \Big\{(a_0, a_1, \ldots, a_t, a_{t+1}) \in [n]^{t+2} \mid a_0, a_1, \ldots, a_{t+1} \text{ are distinct}, \\ (a_i, a_{i-1}) \in E^{(1)}_{(\mathcal{P},\pi)} \forall\, i \in [t+1], \\ (a_i, a_j) \notin E^{(1)}_{(\mathcal{P},\pi)} \forall\, i \in \{2, \ldots, t+1\}, j \leq i-2, \\ C_{a_1,\mathcal{P}} = C_{a_2,\mathcal{P}} = \cdots = C_{a_t,\mathcal{P}}, \\ C_{a_1,\mathcal{P}} \not\leq_\pi C_{a_0,\mathcal{P}},\ C_{a_1,\mathcal{P}} \not\leq_\pi C_{a_{t+1},\mathcal{P}}\Big\}.$$

Also, define $S_5^{(1)} \coloneqq S_4$ and for all $t \in \{2, \ldots, n-2\}$, $S_5^{(t)} \coloneqq \operatorname{argmax}_{(\mathcal{P},\pi) \in S_5^{(t-1)}} \left|D^{(t)}_{(\mathcal{P},\pi)}\right|$.

Then

Condition 5

Proposition

For all $t \in [n-2]$ and $(\mathcal{P}, \pi) \in \mathcal{S}$, let

$$D^{(t)}_{(\mathcal{P},\pi)} := \{(a_0, a_1, \ldots, a_t, a_{t+1}) \in [n]^{t+2} \mid a_0, a_1, \ldots, a_{t+1} \text{ are distinct,}$$
$$(a_i, a_{i-1}) \in E^{(1)}_{(\mathcal{P},\pi)} \forall\, i \in [t+1],$$
$$(a_i, a_j) \notin E^{(1)}_{(\mathcal{P},\pi)} \forall\, i \in \{2, \ldots, t+1\}, j \leq i-2,$$
$$C_{a_1,\mathcal{P}} = C_{a_2,\mathcal{P}} = \cdots = C_{a_t,\mathcal{P}},$$
$$C_{a_1,\mathcal{P}} \not\leq_\pi C_{a_0,\mathcal{P}},\ C_{a_1,\mathcal{P}} \not\leq_\pi C_{a_{t+1},\mathcal{P}}\}.$$

Also, define $S_5^{(1)} := S_4$ and for all $t \in \{2, \ldots, n-2\}$, $S_5^{(t)} := \operatorname{argmax}_{(\mathcal{P},\pi) \in S_5^{(t-1)}} \left| D^{(t)}_{(\mathcal{P},\pi)} \right|$.

Then

- the partially ordered partition associated with $G^\star$ is in $S_5^{(t)}$ for all $t \in [n-2]$, and

Condition 5

Proposition

For all $t \in [n-2]$ and $(\mathcal{P}, \pi) \in \mathcal{S}$, let

$$D^{(t)}_{(\mathcal{P},\pi)} := \{(a_0, a_1, \ldots, a_t, a_{t+1}) \in [n]^{t+2} \mid a_0, a_1, \ldots, a_{t+1} \text{ are distinct}, \\ (a_i, a_{i-1}) \in E^{(1)}_{(\mathcal{P},\pi)} \forall\, i \in [t+1], \\ (a_i, a_j) \notin E^{(1)}_{(\mathcal{P},\pi)} \forall\, i \in \{2, \ldots, t+1\}, j \leq i-2, \\ C_{a_1,\mathcal{P}} = C_{a_2,\mathcal{P}} = \cdots = C_{a_t,\mathcal{P}}, \\ C_{a_1,\mathcal{P}} \not\leq_\pi C_{a_0,\mathcal{P}},\ C_{a_1,\mathcal{P}} \not\leq_\pi C_{a_{t+1},\mathcal{P}}\}.$$

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Then

- the partially ordered partition associated with $G^\star$ is in $S_5^{(t)}$ for all $t \in [n-2]$, and
- for all $t \in [n-2]$ and $(\mathcal{P}, \pi) \in S_5^{(t)}$, $(a_0, a_1, \cdots, a_t, a_{t+1}) \in D^{(t)}_{(\mathcal{P},\pi)}$ if and only if (a_0, a_1, a_2) and (a_{t-1}, a_t, a_{t+1}) are mutually exclusive with respect to the uncovered itinerary $(a_0, a_1, \ldots, a_{t+1})$ in $G^\star$.

Condition 6

If (a_0, a_1, a_2) and (a_{t-1}, a_t, a_{t+1}) are mutually exclusive with respect to some uncovered itinerary $(a_0, a_1, \ldots, a_{t+1})$ and (a_0, b_1, a_{t+1}) is an unshielded imperfect non-conductor (in G_1 and G_2), then a_1 is an ancestor of b in G_1 iff a_1 is an ancestor of b in G_2.

Condition 6

If (a_0, a_1, a_2) and (a_{t-1}, a_t, a_{t+1}) are mutually exclusive with respect to some uncovered itinerary $(a_0, a_1, \ldots, a_{t+1})$ and (a_0, b_1, a_{t+1}) is an unshielded imperfect non-conductor (in G_1 and G_2), then a_1 is an ancestor of b in G_1 iff a_1 is an ancestor of b in G_2.

Proposition

For every $(\mathcal{P}, \pi) \in \mathcal{S}$, let

$$E^{(6)}_{(\mathcal{P},\pi)} := \bigcup_{t=1}^{n-2} \{ ((a_0, a_1, \ldots, a_t, a_{t+1}), (a_0, b, a_{t+1})) \in [n]^{t+5} \mid (a_0, b, a_{t+1}) \in E^{(3)}_{(\mathcal{P},\pi)}, \\ (a_0, a_1, \ldots, a_t, a_{t+1}) \in D^{(t)}_{(\mathcal{P},\pi)}, \\ C_{a_1,\mathcal{P}} \leq_\pi C_{b,\mathcal{P}} \}.$$

Also, define $S_6 := \operatorname{argmin}_{(\mathcal{P},\pi) \in S_5^{(n-2)}} \left| E^{(6)}_{(\mathcal{P},\pi)} \right|$. Then

Condition 6

If (a_0, a_1, a_2) and (a_{t-1}, a_t, a_{t+1}) are mutually exclusive with respect to some uncovered itinerary $(a_0, a_1, \ldots, a_{t+1})$ and (a_0, b_1, a_{t+1}) is an unshielded imperfect non-conductor (in G_1 and G_2), then a_1 is an ancestor of b in G_1 iff a_1 is an ancestor of b in G_2.

Proposition

For every $(\mathcal{P}, \pi) \in \mathcal{S}$, let

$$E^{(6)}_{(\mathcal{P},\pi)} := \bigcup_{t=1}^{n-2} \{ ((a_0, a_1, \ldots, a_t, a_{t+1}), (a_0, b, a_{t+1})) \in [n]^{t+5} \mid (a_0, b, a_{t+1}) \in E^{(3)}_{(\mathcal{P},\pi)}$$

$$(a_0, a_1, \ldots, a_t, a_{t+1}) \in D^{(t)}_{(\mathcal{P},\pi)},$$

$$C_{a_1,\mathcal{P}} \leq_\pi C_{b,\mathcal{P}} \}.$$

Also, define $S_6 := \operatorname{argmin}_{(\mathcal{P},\pi)\in S_5^{(n-2)}} \left| E^{(6)}_{(\mathcal{P},\pi)} \right|$. Then

- the partially ordered partition associated with $G^\star$ is in S_6, and

Condition 6

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Proposition

For every $(\mathcal{P}, \pi) \in \mathcal{S}$, let

$$E^{(6)}_{(\mathcal{P},\pi)} := \bigcup_{t=1}^{n-2} \{ ((a_0, a_1, \ldots, a_t, a_{t+1}), (a_0, b, a_{t+1})) \in [n]^{t+5} \mid (a_0, b, a_{t+1}) \in E^{(3)}_{(\mathcal{P},\pi)}$$

$$(a_0, a_1, \ldots, a_t, a_{t+1}) \in D^{(t)}_{(\mathcal{P},\pi)},$$

$$C_{a_1,\mathcal{P}} \leq_\pi C_{b,\mathcal{P}} \}.$$

Also, define $S_6 := \operatorname{argmin}_{(\mathcal{P},\pi)\in S_5^{(n-2)}} \left| E^{(6)}_{(\mathcal{P},\pi)} \right|$. Then

- the partially ordered partition associated with $G^\star$ is in S_6, and
- for every $(\mathcal{P}, \pi) \in S_6$, $((a_0, a_1, \cdots, a_t, a_{t+1}), (a_0, b, a_{t+1})) \in E^{(6)}_{(\mathcal{P},\pi)}$ if and only if (a_0, a_1, a_2) and (a_{t-1}, a_t, a_{t+1}) are mutually exclusive with respect to the uncovered itinerary $(a_0, a_1, \ldots, a_{t+1})$, (a_0, b_1, a_{t+1}) is an unshielded imperfect non-conductor, and a_1 is an ancestor of b in $G^\star$.

Outline

Greedy optimization algorithm

Algorithm 2 Greedy Markov equivalence class discovery

Input: The set of all the conditional independence statements satisfied by distribution $\mathbb{P}$, two positive integers N and M, and initial partially ordered partitions $(\mathcal{P}_1, \pi_1), \dots, (\mathcal{P}_M, \pi_M)$.

Output: An optimal partially ordered partition.

1: Set $A := \emptyset$.
2: **for** i in $1 : M$ **do**
3:     Set $\hat{\mathcal{P}} := \mathcal{P}_i$ and $\hat{\pi} := \pi_i$.
4:     Perform a depth-first search on $\mathcal{G}$ with root $(\hat{\mathcal{P}}, \hat{\pi})$ to find a directed path from $(\hat{\mathcal{P}}, \hat{\pi})$ to a partially ordered partition $(\tilde{\mathcal{P}}, \tilde{\pi})$ with $\mathrm{GS}(\hat{\mathcal{P}}, \hat{\pi}) > \mathrm{GS}(\tilde{\mathcal{P}}, \tilde{\pi})$. Stop the depth-first search once a directed path of length N consisting of partially ordered partitions of score $\mathrm{GS}(\hat{\mathcal{P}}, \hat{\pi})$ is generated.
5:     **if** $(\tilde{\mathcal{P}}, \tilde{\pi})$ is found **then**
6:         Set $\hat{\mathcal{P}} := \tilde{\mathcal{P}}$ and $\hat{\pi} := \tilde{\pi}$, and go back to step 4.
7:     **else**
8:         Set $A := A \cup \{(\hat{\mathcal{P}}, \hat{\pi})\}$.
9:     **end if**
10: **end for**
11: **return** $\mathrm{argmin}_{(\mathcal{P}, \pi) \in A} \mathrm{GS}(\mathcal{P}, \pi)$.

Outline

FIGURE: Results of running the greedy optimization algorithm on 120 random graphs with different numbers of vertices n where each edge appears in the graph with probability 0.2. Thirty graphs were tested for each n. From left to right, the success rates are 0.93, 0.93, 0.93 and 0.87.

Outline

$$C := \{1, 2, 3, 4, 5, 6\},$$
$$A_C := \{(1, 3), (3, 5), (1, 5), (2, 4), (2, 6), (4, 6), (1, 4), (2, 5)\},$$
$$B_C := \{(7, 1), (8, 1), (7, 4), (8, 4), (9, 6), (9, 2)\},$$
$$ComCh_C := \{(7, 8), (8, 7)\}, \quad NoComCh_C := \{(7, 9), (9, 7)\}.$$

$$\begin{aligned}
&C := \{1,2,3,4,5,6\},\\
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\end{aligned}$$

Two edges $(a,b),(c,b) \in [n] \times C$ are said to be *incompatible* if $a \neq c$ and one of the following happens:

$$C := \{1,2,3,4,5,6\},$$
$$A_C := \{(1,3),(3,5),(1,5),(2,4),(2,6),(4,6),(1,4),(2,5)\},$$
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Two edges $(a,b),(c,b) \in [n] \times C$ are said to be *incompatible* if $a \neq c$ and one of the following happens:

- $a, c \in [n] \setminus C$ and $(a,c) \in NoComCh_C$, or

$$C := \{1,2,3,4,5,6\},$$
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- $a, c \in [n] \setminus C$ and $(a,c) \in NoComCh_C$, or
- $C \cap \{a,c\} \neq \emptyset$, $(a,c) \notin A_C \cup B_C$, and $(c,a) \notin A_C \cup B_C$.

$$
\begin{aligned}
&C := \{1,2,3,4,5,6\},\\
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\end{aligned}
$$

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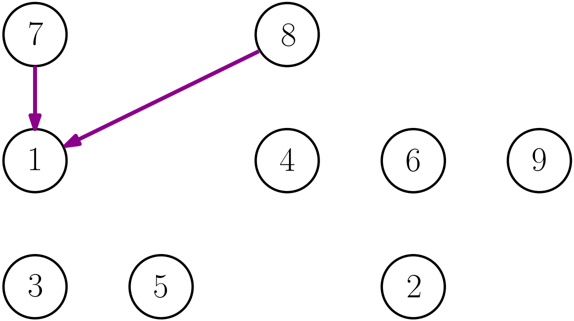
- $a, c \in [n] \setminus C$ and $(a,c) \in NoComCh_C$, or
- $C \cap \{a,c\} \neq \emptyset$, $(a,c) \notin A_C \cup B_C$, and $(c,a) \notin A_C \cup B_C$.

An edge is said to be *safe* to be added to a set of edges if it's not incompatible with any of them.

$$C := \{1, 2, 3, 4, 5, 6\},$$
$$A_C := \{(1, 3), (3, 5), (1, 5), (2, 4), (2, 6), (4, 6), (1, 4), (2, 5)\},$$
$$B_C := \{(7, 1), (8, 1), (7, 4), (8, 4), (9, 6), (9, 2)\},$$
$$ComCh_C := \{(7, 8), (8, 7)\}, \quad NoComCh_C := \{(7, 9), (9, 7)\}.$$

First assigns a common child to each pair in $ComCh_C$.



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$$A_C := \{(1,3),(3,5),(1,5),(2,4),(2,6),(4,6),(1,4),(2,5)\},$$
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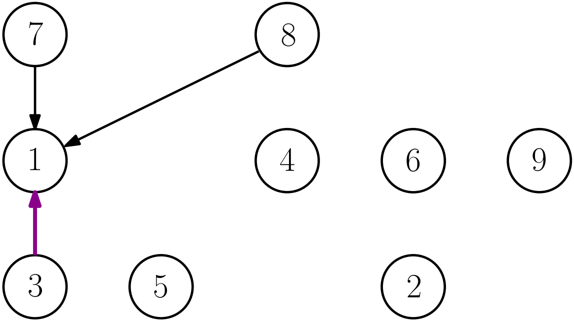
Starts from a vertex in C with the lowest degree in $([n], A_C \cup B_C)$.



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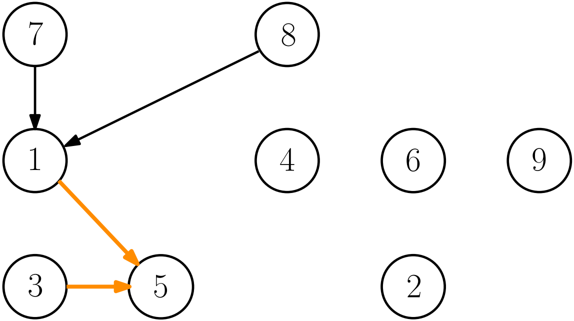
When an edge is added, the algorithm makes sure it's safe. If not, the algorithm initiates a correction process. In this process, the algorithm first tries to remove the unsafe edge.



An SCCR algorithm: Construct and correct

$$C := \{1,2,3,4,5,6\},$$
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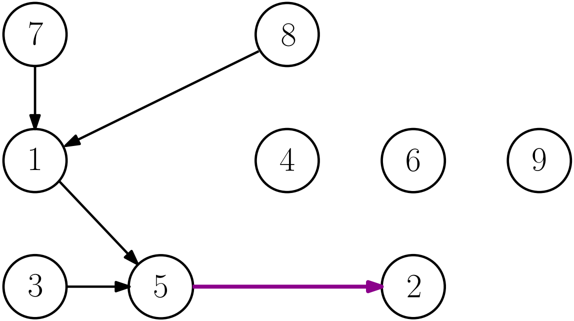
In each iteration, the algorithm adds an edge from $A_C \cup B_C$ to the construction such that the added edges form an *almost* directed path.



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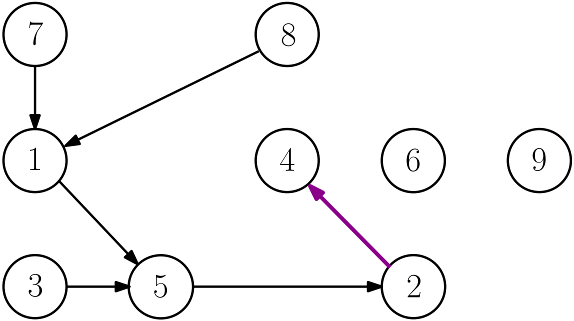

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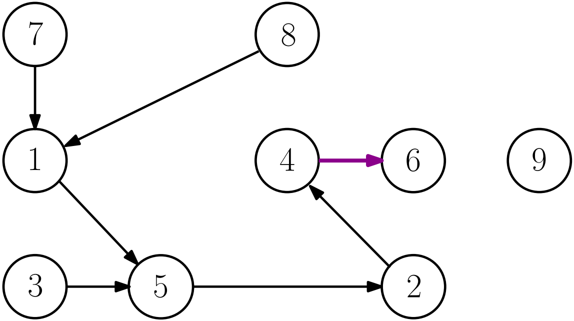

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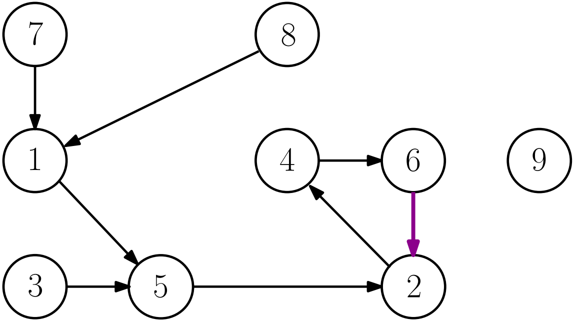
Unsafe edge added (incompatible with $(5, 2)$)! Correction process starts.



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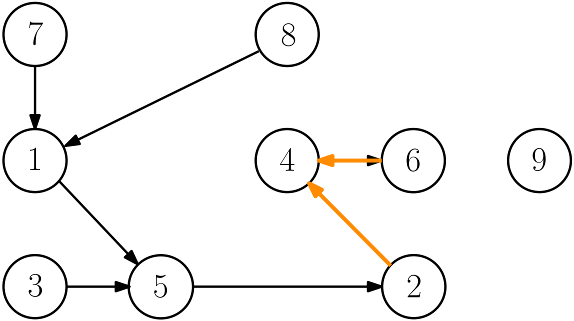
Problem resolved!



An SCCR algorithm: Construct and correct

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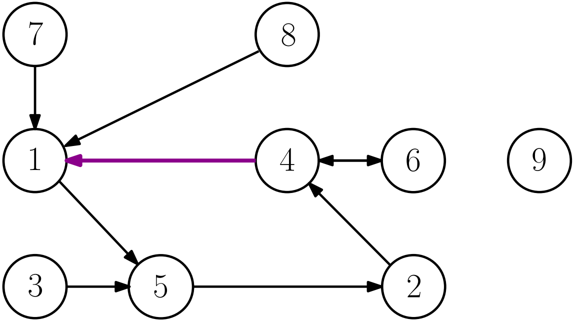

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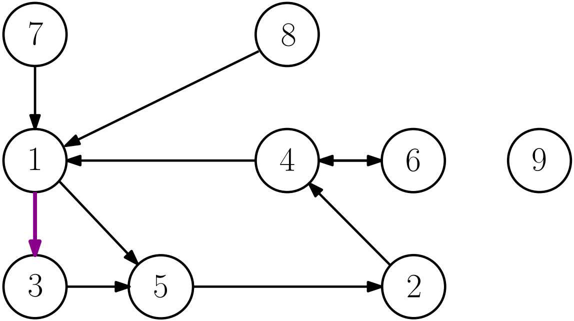

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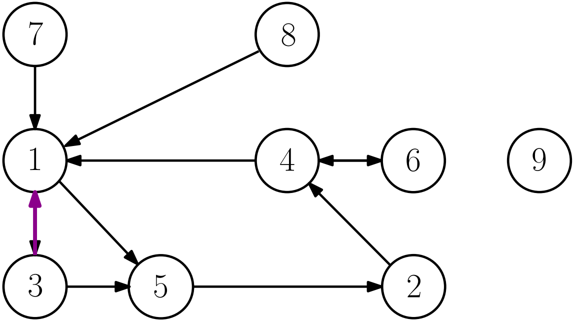
Unsafe edge added (incompatible with $(7,1)$)! Correction process starts.



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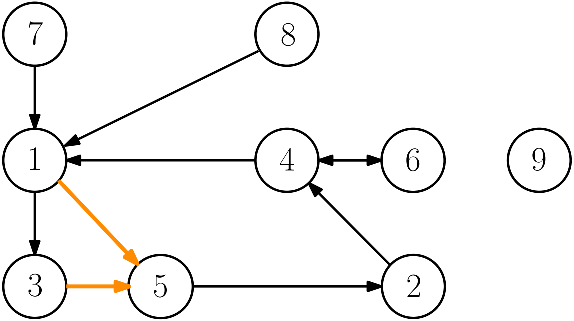
Problem resolved!



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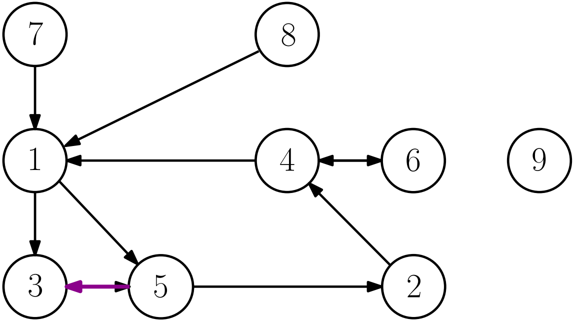

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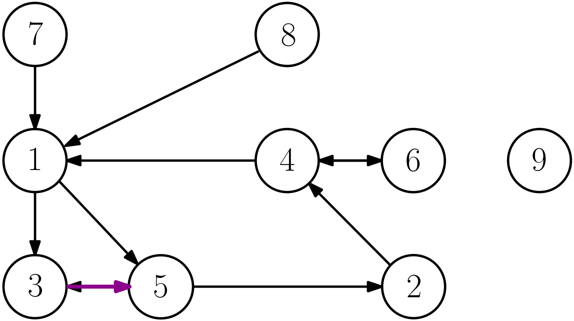

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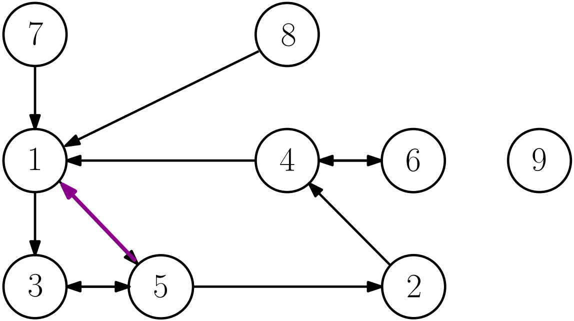
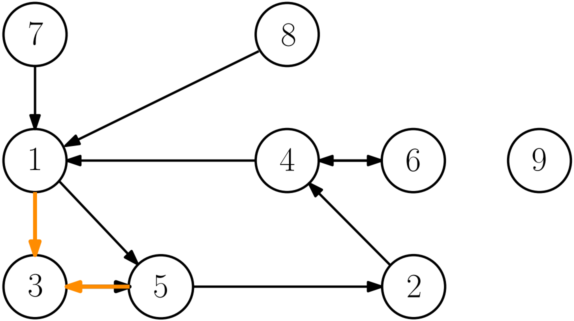
$$ComCh_C := \{(7,8), (8,7)\}, \quad NoComCh_C := \{(7,9), (9,7)\}.$$

Unsafe edge added (incompatible with $(7,1)$)! Correction process starts.



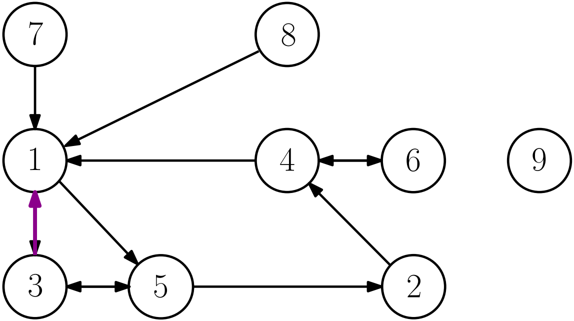
$$C := \{1, 2, 3, 4, 5, 6\},$$
$$A_C := \{(1,3), (3,5), (1,5), (2,4), (2,6), (4,6), (1,4), (2,5)\},$$
$$B_C := \{(7,1), (8,1), (7,4), (8,4), (9,6), (9,2)\},$$
$$ComCh_C := \{(7,8), (8,7)\}, \quad NoComCh_C := \{(7,9), (9,7)\}.$$

Problem resolved!



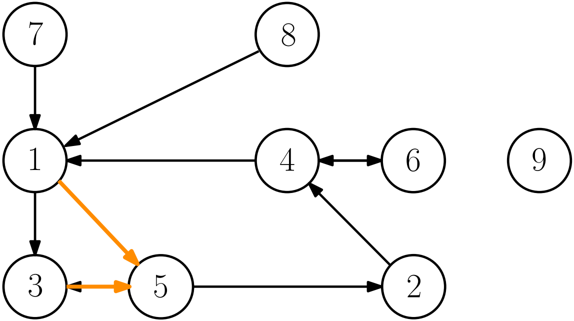
$C := \{1, 2, 3, 4, 5, 6\},$
$A_C := \{(1, 3), (3, 5), (1, 5), (2, 4), (2, 6), (4, 6), (1, 4), (2, 5)\},$
$B_C := \{(7, 1), (8, 1), (7, 4), (8, 4), (9, 6), (9, 2)\},$
$ComCh_C := \{(7, 8), (8, 7)\}, \quad NoComCh_C := \{(7, 9), (9, 7)\}.$

Unsafe edge added
(incompatible with $(7, 1)$)!
Correction process starts.



$$C := \{1,2,3,4,5,6\},$$
$$A_C := \{(1,3),(3,5),(1,5),(2,4),(2,6),(4,6),(1,4),(2,5)\},$$
$$B_C := \{(7,1),(8,1),(7,4),(8,4),(9,6),(9,2)\},$$
$$ComCh_C := \{(7,8),(8,7)\}, \quad NoComCh_C := \{(7,9),(9,7)\}.$$

Problem resolved!



$C := \{1, 2, 3, 4, 5, 6\},$

$A_C := \{(1,3), (3,5), (1,5), (2,4), (2,6), (4,6), (1,4), (2,5)\},$

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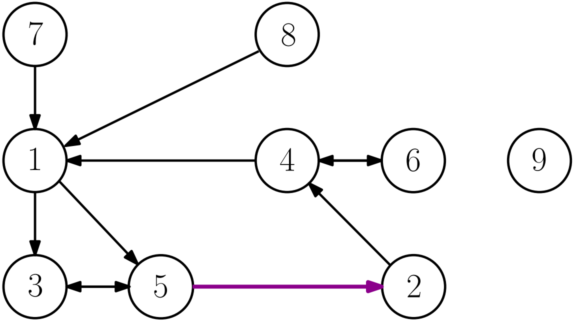
$ComCh_C := \{(7,8), (8,7)\}, \quad NoComCh_C := \{(7,9), (9,7)\}.$



$$C := \{1,2,3,4,5,6\},$$
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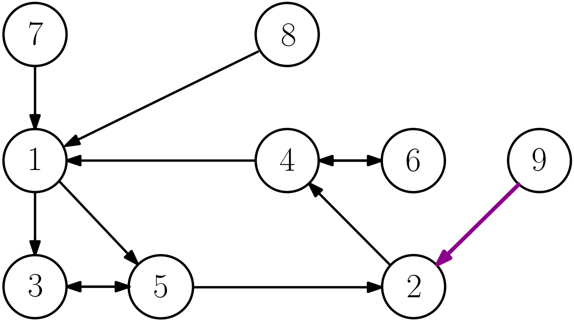
$$B_C := \{(7,1),(8,1),(7,4),(8,4),(9,6),(9,2)\},$$
$$ComCh_C := \{(7,8),(8,7)\}, \quad NoComCh_C := \{(7,9),(9,7)\}.$$

Unsafe edge added (incompatible with $(5,2)$)! Correction process starts.



$$C := \{1, 2, 3, 4, 5, 6\},$$
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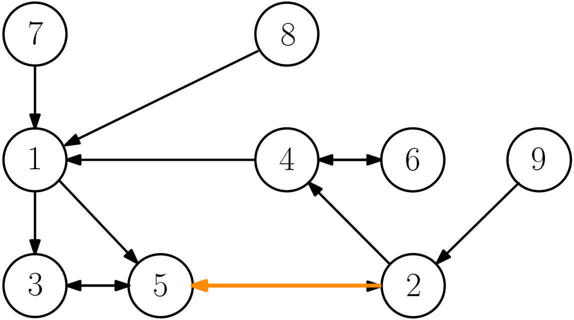
$$ComCh_C := \{(7,8), (8,7)\}, \quad NoComCh_C := \{(7,9), (9,7)\}.$$

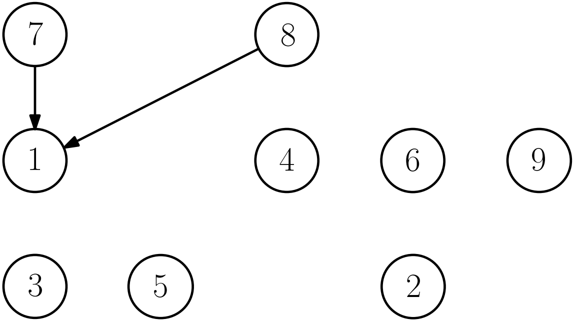
In the correction process, when removing fails, the algorithm flips the edge (in case the head and the tail are both in C). Each edge is allowed to be corrected at most once. So, if the flipped edge is incompatible with an edge already affected in a correction process (in this case, $(1,5)$), the algorithm jumps back to the stage right before the iteration involving the correction of this edge started.



An SCCR algorithm: Construct and correct

$$C := \{1,2,3,4,5,6\},$$
$$A_C := \{(1,3),(3,5),(1,5),(2,4),(2,6),(4,6),(1,4),(2,5)\},$$
$$B_C := \{(7,1),(8,1),(7,4),(8,4),(9,6),(9,2)\},$$
$$ComCh_C := \{(7,8),(8,7)\}, \quad NoComCh_C := \{(7,9),(9,7)\}.$$

The algorithm is only allowed to erase part of its progress N times.
If that happens, the algorithm shuffles A_C and B_C and starts over avoiding the choice leading to its first failed attempt.



$$C := \{1,2,3,4,5,6\},$$
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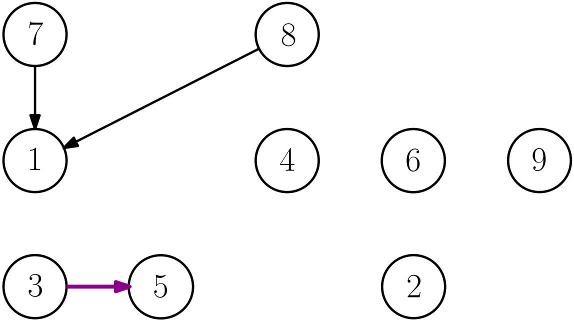

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$$C := \{1, 2, 3, 4, 5, 6\},$$
$$A_C := \{(1,3), (3,5), (1,5), (2,4), (2,6), (4,6), (1,4), (2,5)\},$$
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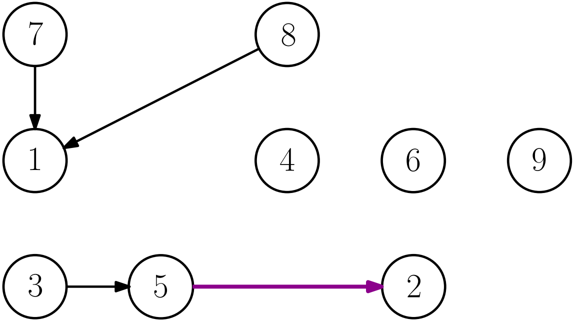

$$C := \{1,2,3,4,5,6\},$$
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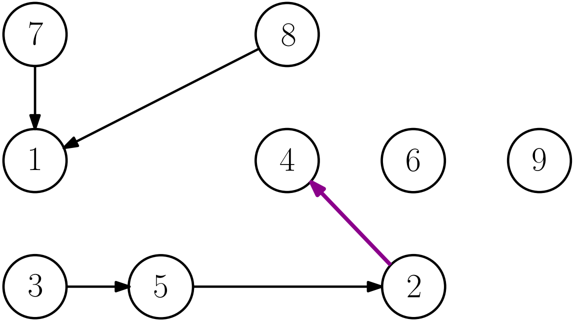

$$C := \{1, 2, 3, 4, 5, 6\},$$
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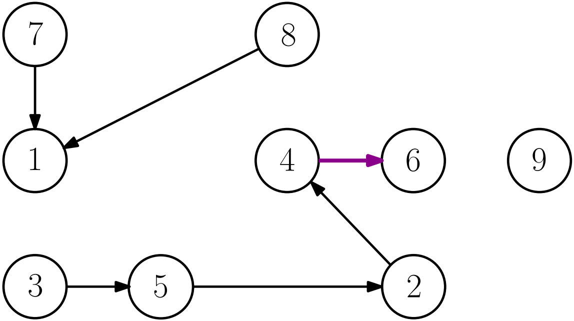
$$ComCh_C := \{(7,8), (8,7)\}, \quad NoComCh_C := \{(7,9), (9,7)\}.$$



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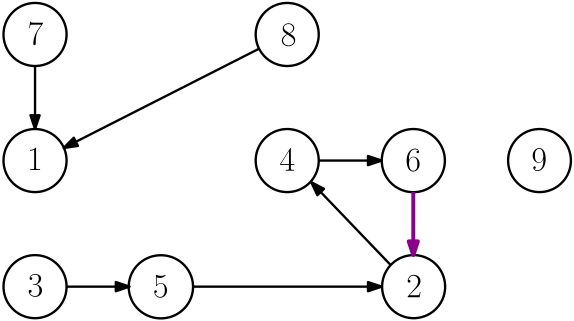
$$ComCh_C := \{(7,8),(8,7)\}, \quad NoComCh_C := \{(7,9),(9,7)\}.$$

Unsafe edge (incompatible with $(5,2)$)! Correction process starts.



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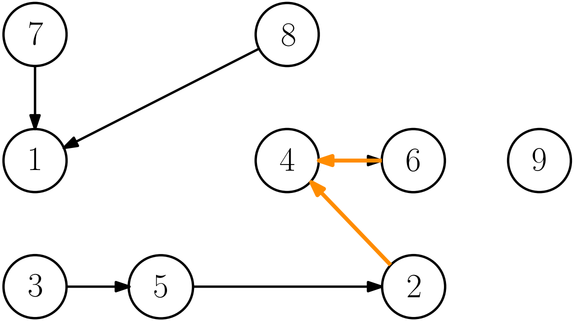
$ComCh_C := \{(7,8), (8,7)\}, \quad NoComCh_C := \{(7,9), (9,7)\}.$

Problem resolved!



An SCCR algorithm: Construct and correct

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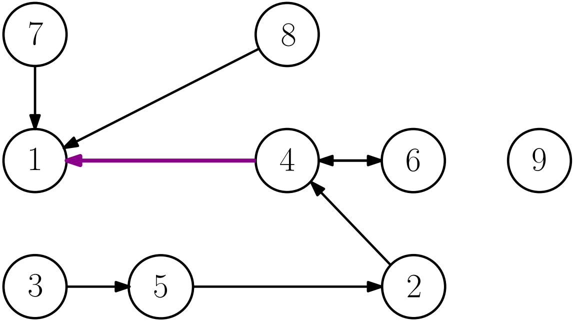

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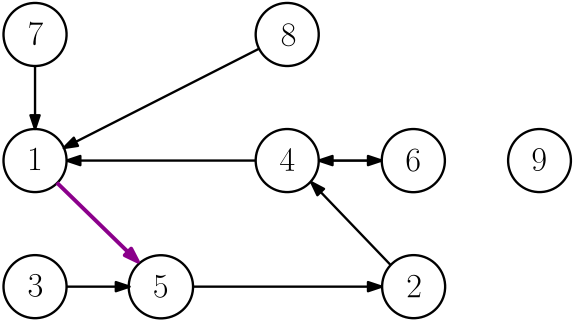

$$C := \{1, 2, 3, 4, 5, 6\},$$
$$A_C := \{(1, 3), (3, 5), (1, 5), (2, 4), (2, 6), (4, 6), (1, 4), (2, 5)\},$$
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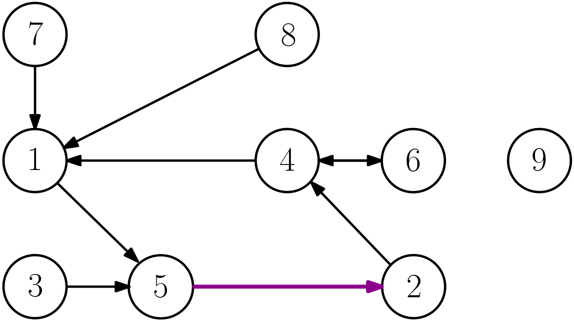
$$ComCh_C := \{(7, 8), (8, 7)\}, \quad NoComCh_C := \{(7, 9), (9, 7)\}.$$



$$C := \{1, 2, 3, 4, 5, 6\},$$
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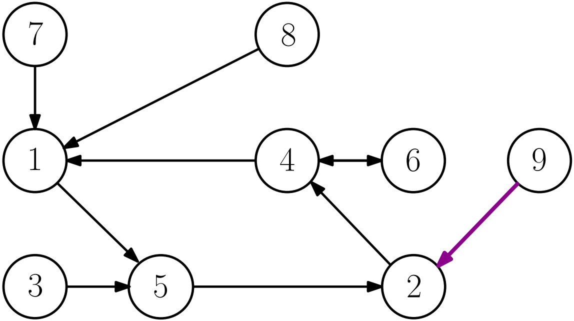
$$ComCh_C := \{(7,8), (8,7)\}, \quad NoComCh_C := \{(7,9), (9,7)\}.$$

Unsafe edge added
(incompatible with $(5,2)$)!
Correction process starts.



$$C := \{1,2,3,4,5,6\},$$
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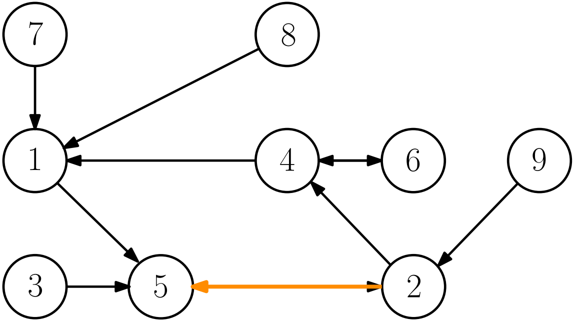
$$ComCh_C := \{(7,8),(8,7)\}, \quad NoComCh_C := \{(7,9),(9,7)\}.$$

Failed to remove $(9,2)$.
So, moved to $(5,2)$.
Failed to remove $(5,2)$.
So, flips $(5,2)$.
But now $(2,5)$ is incompatible with $(1,5)$.



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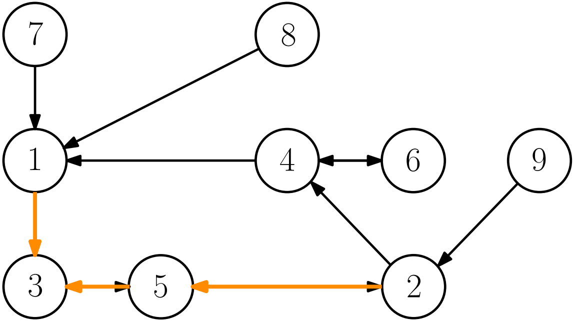
$$ComCh_C := \{(7,8), (8,7)\}, \quad NoComCh_C := \{(7,9), (9,7)\}.$$

$(9, 2)$ is still incompatible with another $(5, 2)$ in the construction. So, the algorithm jumps back to the stage right before adding that $(5, 2)$.



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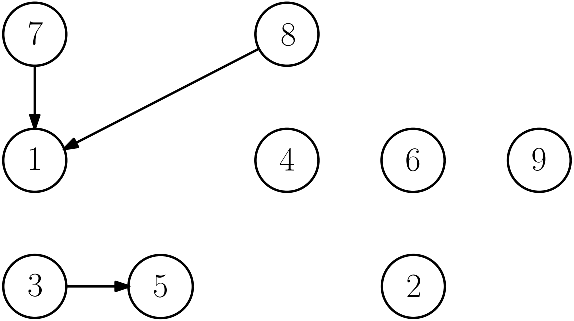
Shuffles A_C and B_C and starts over.



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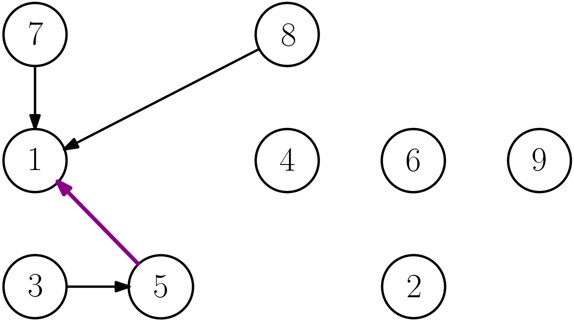
$$ComCh_C := \{(7,8),(8,7)\}, \quad NoComCh_C := \{(7,9),(9,7)\}.$$

Unsafe edge (incompatible with $(7,1)$)! Correction process starts.



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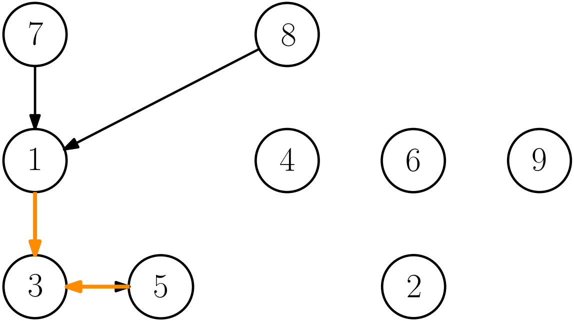
$$ComCh_C := \{(7,8),(8,7)\}, \quad NoComCh_C := \{(7,9),(9,7)\}.$$

Problem resolved!



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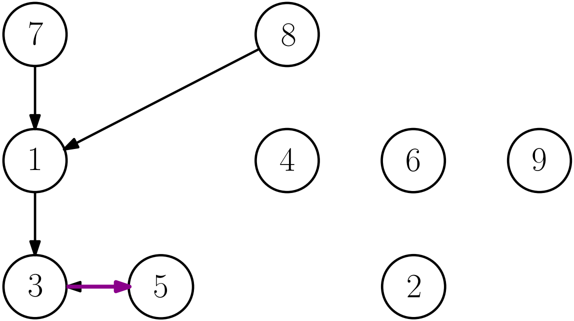

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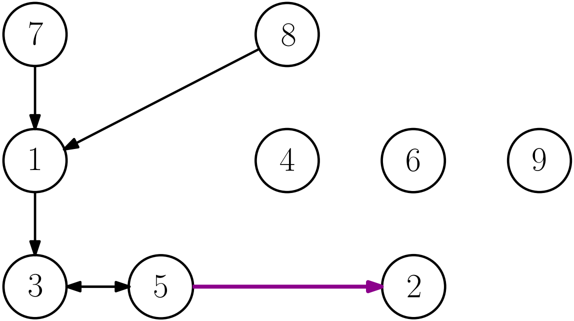

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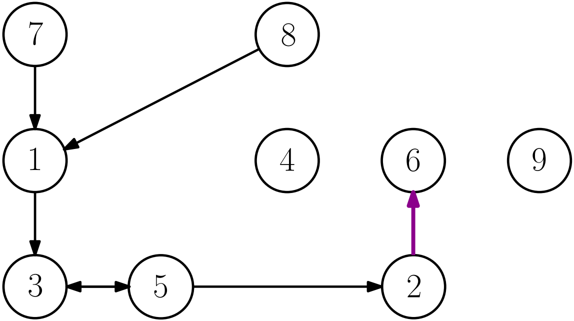

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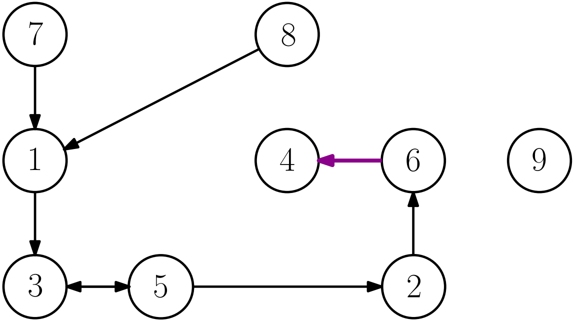

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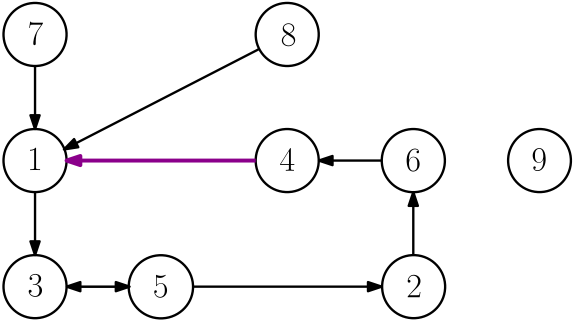

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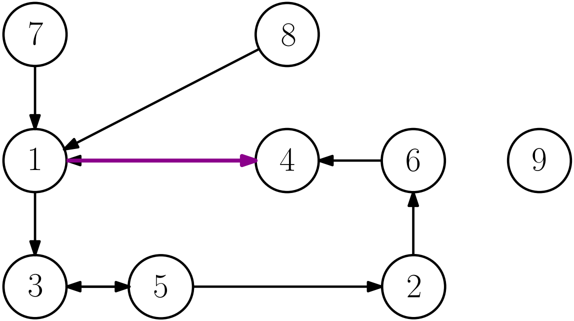
Unsafe edge (incompatible with $(6,4)$)! Correction process starts.



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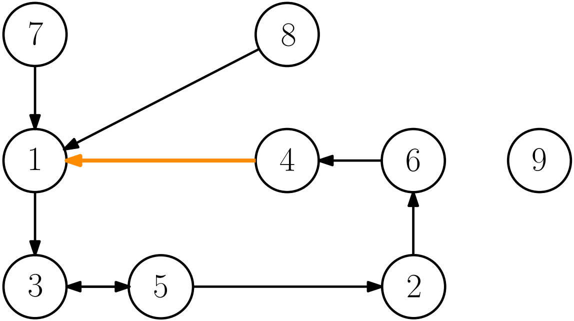
$$B_C := \{(7,1), (8,1), (7,4), (8,4), (9,6), (9,2)\},$$
$$ComCh_C := \{(7,8), (8,7)\}, \quad NoComCh_C := \{(7,9), (9,7)\}.$$

Failed to remove $(1, 4)$. So, flips $(1, 4)$ and now the problem is resolved!



$C := \{1, 2, 3, 4, 5, 6\},$
$A_C := \{(1,3), (3,5), (1,5), (2,4), (2,6), (4,6), (1,4), (2,5)\},$
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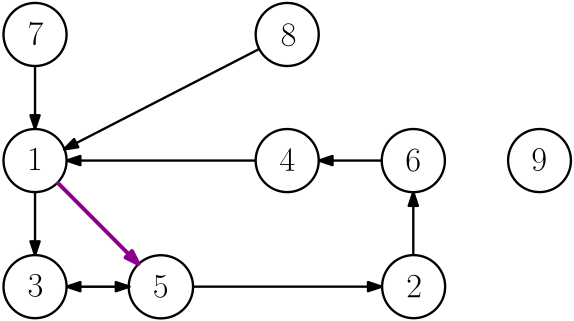

An SCCR algorithm: Construct and correct

$$C := \{1, 2, 3, 4, 5, 6\},$$
$$A_C := \{(1, 3), (3, 5), (1, 5), (2, 4), (2, 6), (4, 6), (1, 4), (2, 5)\},$$
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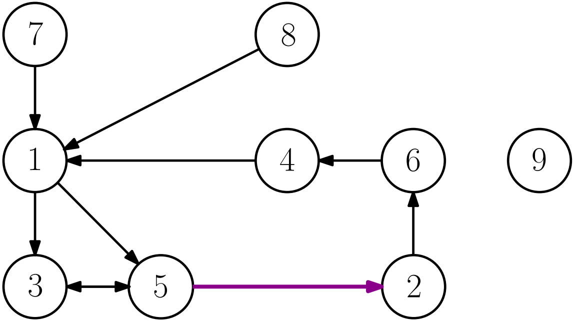

An SCCR algorithm: Construct and correct

$$C := \{1,2,3,4,5,6\},$$
$$A_C := \{(1,3),(3,5),(1,5),(2,4),(2,6),(4,6),(1,4),(2,5)\},$$
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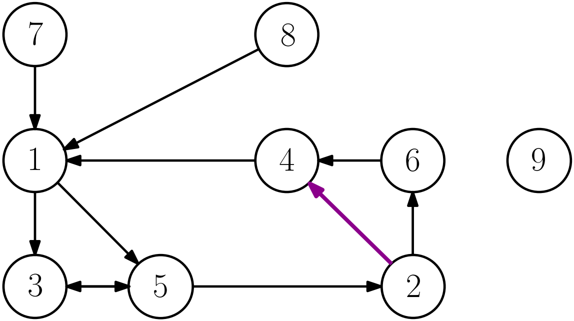

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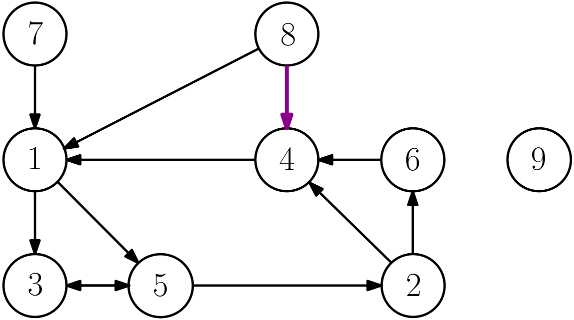
Unsafe edge (incompatible with $(2,4)$)! Correction process starts.



$$C := \{1,2,3,4,5,6\},$$
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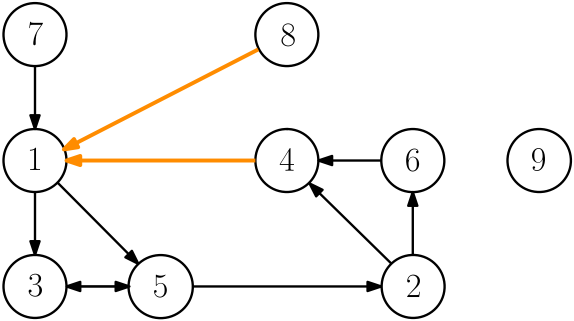
Problem resolved!



$C := \{1, 2, 3, 4, 5, 6\},$
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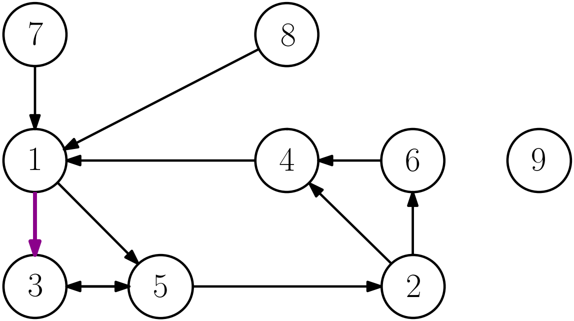

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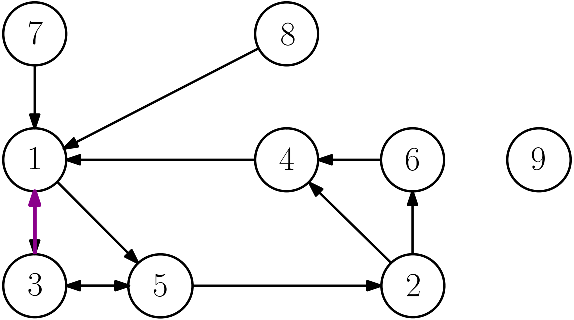
Unsafe edge (incompatible with $(7,1)$)! Correction process starts.



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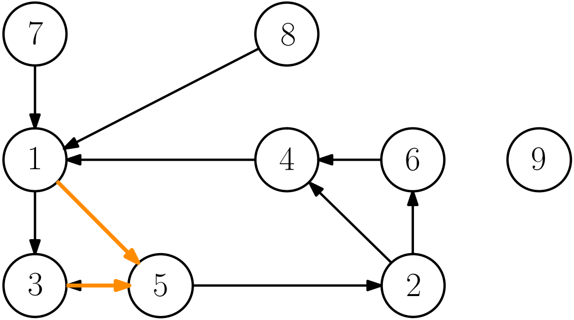
Problem resolved!



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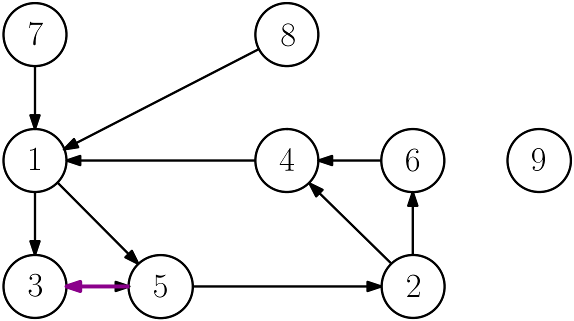

$$C := \{1,2,3,4,5,6\},$$
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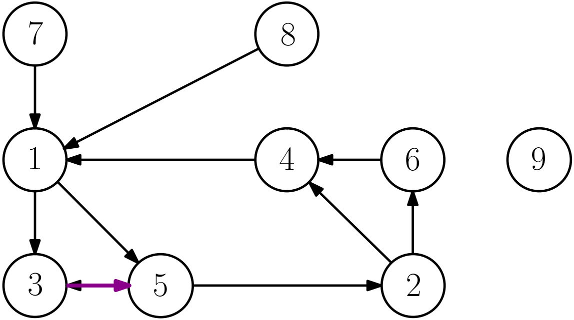

An SCCR algorithm: Construct and correct

$$C := \{1, 2, 3, 4, 5, 6\},$$
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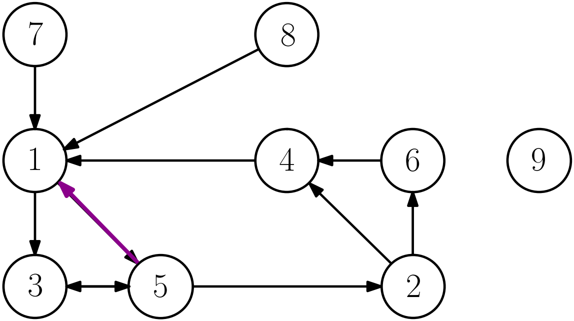
Unsafe edge (incompatible with $(7,1)$)! Correction process starts.



$$C := \{1, 2, 3, 4, 5, 6\},$$
$$A_C := \{(1,3), (3,5), (1,5), (2,4), (2,6), (4,6), (1,4), (2,5)\},$$
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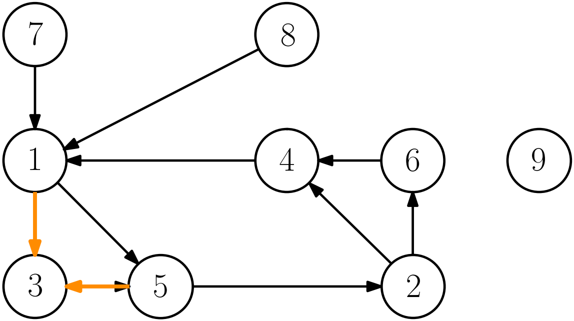
$$ComCh_C := \{(7,8), (8,7)\}, \quad NoComCh_C := \{(7,9), (9,7)\}.$$

Problem resolved!



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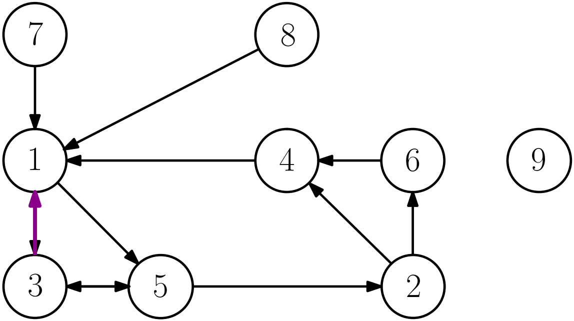
Unsafe edge (incompatible with $(7, 1)$)! Correction process starts.



$$C := \{1, 2, 3, 4, 5, 6\},$$
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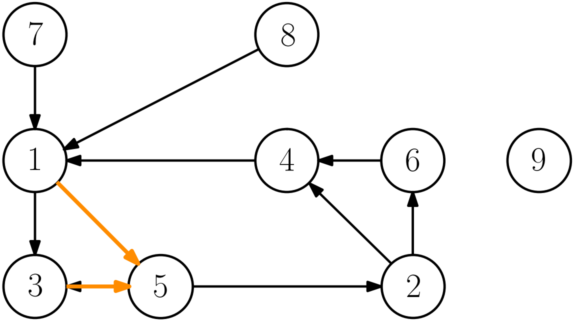
Unsafe edge (incompatible with $(7, 1)$)! Correction process starts.



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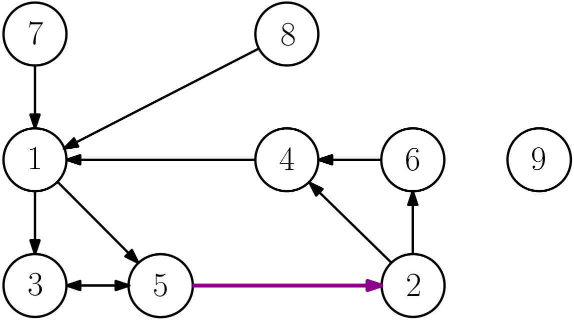

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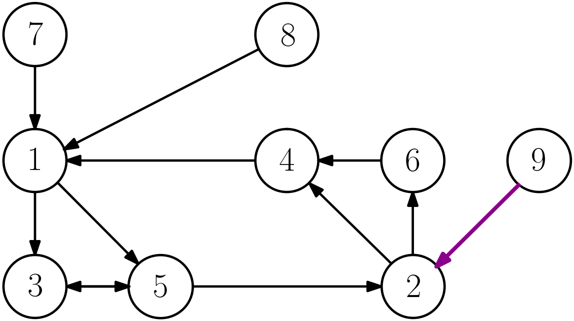
Unsafe edge (incompatible with $(5,2)$)! Correction process starts.



$$C := \{1, 2, 3, 4, 5, 6\},$$
$$A_C := \{(1,3), (3,5), (1,5), (2,4), (2,6), (4,6), (1,4), (2,5)\},$$
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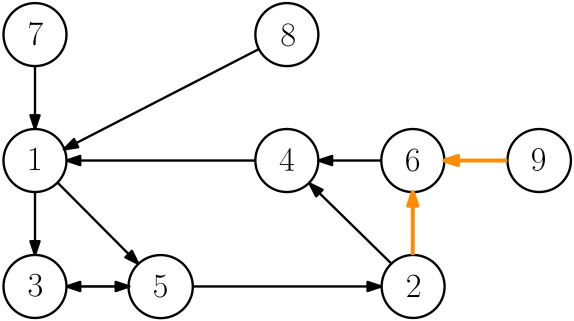
Problem resolved!



$$C := \{1, 2, 3, 4, 5, 6\},$$
$$A_C := \{(1,3), (3,5), (1,5), (2,4), (2,6), (4,6), (1,4), (2,5)\},$$
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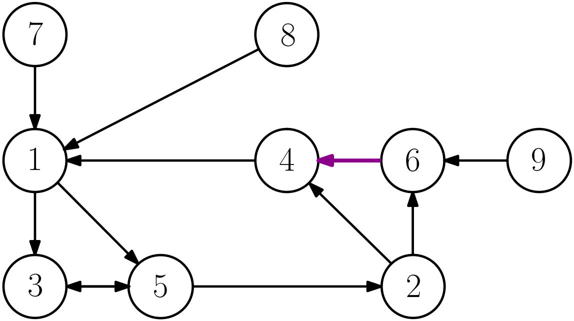

$$C := \{1, 2, 3, 4, 5, 6\},$$
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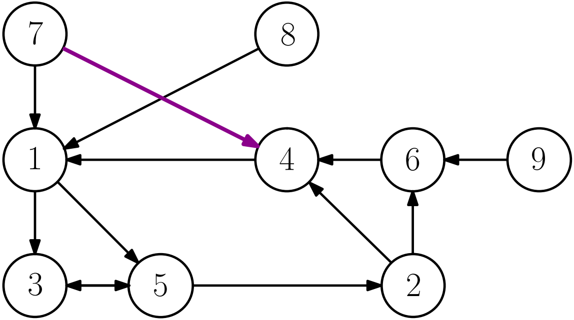
Unsafe edge (incompatible with $(2,4)$)! Correction process starts.



$$C := \{1,2,3,4,5,6\},$$
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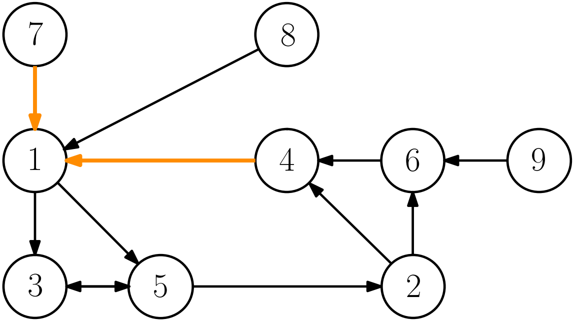
$$ComCh_C := \{(7,8),(8,7)\}, \quad NoComCh_C := \{(7,9),(9,7)\}.$$

Problem resolved!
The algorithm now outputs this construction.



Correctness of the SCCR algorithm

Proposition

Suppose a subset $C \subseteq [n]$ and sets $A_C \subseteq C^2$, $B_C \subseteq ([n] \setminus C) \times C$, $ComCh_C \subseteq ([n] \setminus C)^2$ and $NoComCh_C \subseteq ([n] \setminus C)^2$ are given. For any $N \in \mathbb{N}$, if the SCCR algorithm outputs a set $E_C \subseteq [n]^2$, then E_C satisfies the desired properties.

Proposition

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Conjecture

For all $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ that satisfies the following: Let $(\mathcal{P}_0, \pi_0)$ be the partially ordered partition associated with $G^\star$. Suppose $C \in \mathcal{P}_0$ and

$$A_C := \left\{ (a,b) \in E^{(1)}_{(\mathcal{P}_0, \pi_0)} \;\middle|\; a, b \in C \right\},$$

$$B_C := \left\{ (a,b) \in E^{(1)}_{(\mathcal{P}_0, \pi_0)} \;\middle|\; a \notin C, b \in C, C_{a,\mathcal{P}_0} \leq_{\pi_0} C_{b,\mathcal{P}_0} \right\}.$$

Moreover, let $ComCh_C$ and $NoComCh_C$ be the sets of pairs that must have and must not have common children in C per conditions of the previously stated theorem. Then given $C, A_C, B_C, ComCh_C, NoComCh_C$ and N, with high probability the SCCR algorithm outputs a set $E_C \subseteq [n]^2$ satisfying the desired properties.

Proposition

Suppose a subset $C \subseteq [n]$ and sets $A_C \subseteq C^2$, $B_C \subseteq ([n] \setminus C) \times C$, $ComCh_C \subseteq ([n] \setminus C)^2$ and $NoComCh_C \subseteq ([n] \setminus C)^2$ are given. For any $N \in \mathbb{N}$, if the SCCR algorithm outputs a set $E_C \subseteq [n]^2$, then E_C satisfies the desired properties.

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Moreover, let $ComCh_C$ and $NoComCh_C$ be the sets of pairs that must have and must not have common children in C per conditions of the previously stated theorem. Then given $C, A_C, B_C, ComCh_C, NoComCh_C$ and N, with high probability the SCCR algorithm outputs a set $E_C \subseteq [n]^2$ satisfying the desired properties.

Back to simulations